Quantum coding with low-depth random circuits

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Random quantum circuits have played a central role in establishing the computational advantages of near-term quantum computers over their conventional counterparts [1]. Despite remarkable progress in quantum control and measurement across different quantum computing platforms [1–7], it remains a major challenge to develop error correction strategies that are practically relevant for near-term applications of quantum computers [8, 9]. Many of the most well-studied codes for near-term devices require either a high encoding circuit depth or, at low depth, achieve only a low coding rate (number of logical qubits/physical qubits) or require all-to-all connectivity [10]. In this work, we show that a spatially local random shallow circuit of logarithmic depth or better can generate high-performing codes for local noise models, even at very large coding rates.

Our approach to quantum code design is rooted in arguments from statistical physics and establishes several deep connections between quantum coding theory and critical phenomena in phase transitions. In addition, we introduce a method of targeted measurements to significantly improve random coding performance. These latter results provide interesting connections to the emerging topic of measurement-induced quantum phase transitions [11–17].

At a fundamental level, the study of random error correcting codes has a long history that dates back to the foundational work on information theory by Shannon [18], who showed that random classical codes achieve the capacity of many local error models. Moreover, following early pioneering studies by Gallager [19, 20], random constructions of classical linear codes (e.g., the random LDPC code) have even been found to be practically relevant for designing coding strategies for digital communication. Random coding results in the quantum case have only recently been investigated [21–24], but these groundbreaking works focused on the depth required to achieve a very strong form of quantum error correction related to the decoupling theorem.

In this submission, we study the circuit depth required to achieve zero failure probability for the specific local error model of erasures. In the case of this weaker metric, that is also a practically relevant metric for experiments [25, 26], we are able to improve upon the previous bounds to achieve high-performance coding with random circuits in D spatial dimensions from depth $O(N^{1/D})$ to depth $O(\log N)$. Furthermore, by using quantum measurements to remove low-quality logical operators from the code, we introduce a novel "expurgation" algorithm to achieve high-performance coding at sub-logarithmic depths above one dimension ($D > 1$).

To validate our theoretical approach, we undertake a thorough study of the depth requirements to achieve the quantum channel capacity of the erasure channel, which also points to several deep connections between error correction thresholds and critical phenomena in phase transitions. In our approach, we use the critical finite-size scaling properties of the erasure threshold for the optimal codes as a benchmark performance metric for our low-depth random codes. In $D = 1$, the requisite depth to approach capacity scales as $O(\sqrt{N})$, which is derived by mapping the threshold problem to a problem in the statistics of random walks. Above one dimension, we find that the circuit depth required to achieve zero failure probability below capacity does not change upon approaching the capacity (the so-called moderate deviation limit in coding theory). The theoretical basis for this result can be traced back to fundamental Imry-Ma arguments from statistical physics about the relevance of spatial randomness to the critical behavior of thermal phase transitions [27].

More specifically, we consider stabilizer codes generated by two-local random Clifford circuits on hypercubic lattices in D spatial dimensions or on all-to-all coupled networks. For concreteness, we consider codes of rate $R = 1/2$ in which every other qubit is mapped to an encoded or "logical"

D	$e < e_c$	$e = e_c - O(\frac{1}{N^b})$	Expur. $e < e_c$
1	$(\frac{1}{e_c - e}) \log N$	$N^{1/2}$	$\log N$
2	$\log N$	$\log N$ (conj.)	$(\log N)^c$, $c < 1$
> 2	$\log N$	$\log N$	$(\log N)^c$, $c < 1$

TABLE I. Random Clifford circuit depths required to reach zero failure probability for finite-rate codes under erasure errors in D dimensions. Here, $e_c - O(1/N^b)$ denotes coding arbitrarily close to the critical region of the erasure threshold in the thermodynamic limit. We find $b = 1$ for the fixed-fraction erasure model. $D = 2$ is the marginal dimension for the relevance of spatial randomness in the errors to the threshold behavior, which makes the scaling at capacity difficult to reliably determine from numerics or Imry-Ma arguments. The last column shows the results upon expurgation of bad logical operators using quantum measurements.

qubit. We provide a summary of the main results found in this work in Table I. The error model is taken to be an erasure model where eN sites of an N-qubit system are randomly selected and traced out of the system, with those sites heralded to the decoder but unknown to the encoder.

The main results we will highlight in our talk are in two dimensions ($D = 2$). In order for the code to correct a given region with an excess number of erasures, there needs to be a sufficiently large number of syndromes to identify the possible errors in that region. Notably, for a depth d random circuit the random fluctuations in erasure number within a given region can be overcome by the overlapping syndromes near the boundary of that region whenever

$$L^{D-1} d \sim \sqrt{N} \sim L^{D/2} \to d \sim L^{1-D/2}. \tag{1}$$

This tension between random fluctuations and ordering tendencies is familiar from Imry-Ma arguments. This scaling indicates that $D = 2$ is the marginal dimension for the relevance of random erasure locations. In Fig. 1(a), we show the numerical results for the recovery probability through the erasure threshold at different values of the depth. We clearly see the exponential convergence to the infinite-depth limit throughout the critical region.

To further improve the scaling with depth, we first note that the dominant failure mode at depths $[\log N]^{1/D} \leq d \leq \log N$ are rare regions with bad logical qubits. In our expurgation algorithm, we essentially remove these rare logical operators from the code. Specifically, we begin with a stabilizer code with many logical qubits. We then randomly generate an erasure pattern $\boldsymbol{e}$. Using row reduction, we can form a basis of linearly independent uncorrectable Pauli errors, i.e., these errors map to the all-zero syndrome, but have nontrivial logical operator content. We then perform a sequence of projective measurements of these operators on the code space density matrix to form a new stabilizer or subsystem code. This procedure is iterated many times until either the rate of the code approaches a specified target value, the failure probability reaches a certain threshold, or the number of logical operators goes to zero (i.e., expurgation fails).

In Fig. 1(b), we provide an illustrative example of the performance improvements that are possible with this expurgation strategy for 2D random circuit encodings. Here, d^* is the interpolated depth required to reach a 50% failure probability. We see nearly linear scaling of d^* with $\log N$ before expurgation. After expurgation, d^* has a strongly sublinear scaling with $\log N$. All expurgated logicals were turned into gauge qubits in this example, which has the advantage that the syndrome check operators are unchanged and can retain a low-weight.

To summarize, we study quantum error correcting codes generated by low depth random circuits. In any spatial dimension, we find that a depth $O(\log N)$ random circuit is necessary and sufficient to achieve good coding against erasure errors below the channel capacity. However, in 1D, coding arbitrarily close to capacity requires a depth $O(\sqrt{N})$ circuit due to the relevance of spatial randomness in errors at threshold. The marginal dimension for high-performance low-depth coding at capacity is 2D where spatial randomness becomes an irrelevant perturbation at threshold.

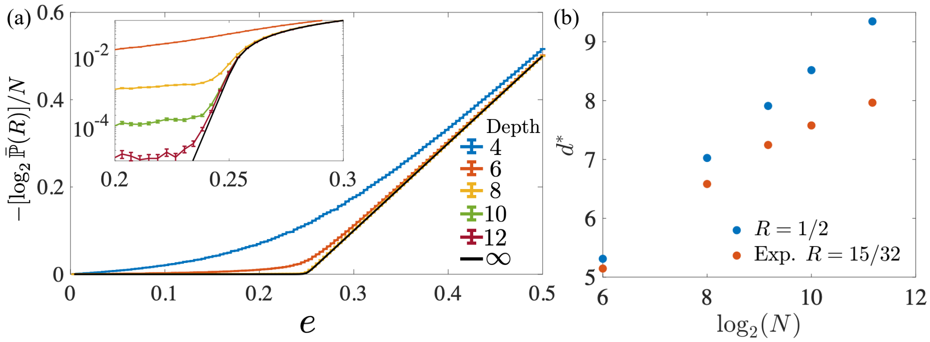

FIG. 1. (a) Code-averaged recovery probability $\bar{\mathbb{P}}(R)$ vs erasure fraction for a two-dimensional random circuit in a brickwork arrangement of gates with periodic boundary conditions for different depths d for $N = 256$ and rate $R = 1/2$. We sequentially cycle through 4 layers so that each site interacts with its north, east, south, west neighbor for each 4 units of depth. The inset shows the same data on a logarithmic scale, illustrating the exponentially scaling with depth $\mathbb{P}(F) \sim e^{-d/A}$ for $e < e_c$. Each two qubit gate in the circuit consists of an iSWAP gate followed by a random single-site Clifford on each site. (b) Interpolated depth d^* to reach 50 % failure probability for a random 2D circuit vs log-system size. Each two qubit gate in the circuit is a random Clifford gate. All expurgated logical operators were turned into gauge operators. We removed only $N/32$ logicals to aid in extracting the scaling vs $\log_2 N$ to large sizes and small depths. We took an erasure fraction $n_e/N = 1/8$ for the expurgation and the calculation of the failure probability.

Although spatial randomness in the errors becomes irrelevant above 1D, there are still large inhomogeneities in the quality of the random code due to random circuit fluctuations. We showed that the effects of code randomness in $D > 1$ can be mitigated by expurgating low-weight logical operators from the code using quantum measurements. With this method, we found that high-performance coding becomes possible at sub-log-N depths. Codes with rates near $1/2$ generated by our random coding algorithms can achieve high performance at depth 4–8 in 2D for large erasure rates and block sizes of thousands of qubits.

The results in this work open up many directions for future research. To develop these codes for use on near-term devices, a more general theory of optimal decoding for Pauli error channels should be developed. Efficient optimal decoding can likely be implemented for these low-depth codes by taking advantage of their strongly local nature and log-depth encodings. It will also be interesting to consider the performance of these codes in conventional threshold theorems, including strategies for achieving full fault-tolerance.

Moreover, it has now been well established that fault-tolerant thresholds can be significantly improved by tailoring codes to the detailed properties of the noise [28–31]. The expurgation algorithm provides a wide variety of additional techniques to tailor codes to specific noise models. In addition, it may be possible to further improve the expurgation by using quantum measurements that explicitly implement entanglement swapping, similar to techniques used for the measurement based preparation of the surface code states [32, 33].

In conclusion, we believe our work adds crucial evidence to an emerging consensus in quantum information science that quantum information processing with error correction may not be that far from practical experimental reality. The introduction of concrete evaluation metrics and practical algorithms to generate high-performing codes has the potential to be a boon to the field of quantum error mitigation strategies in near-term devices that rely on quantum error correcting codes. For these reasons, we believe these results will be a valuable contribution to QIP.

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