Quantum Coding with Low-Depth Random Circuits

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References and Collaborations

Stefan Krastanov

David A. Huse

Liang Jiang

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arXiv:2010.09775

Discussions: Steve Girvin, Pradeep Niroula, Sarang Gopalakrishnan

Random Classical Coding

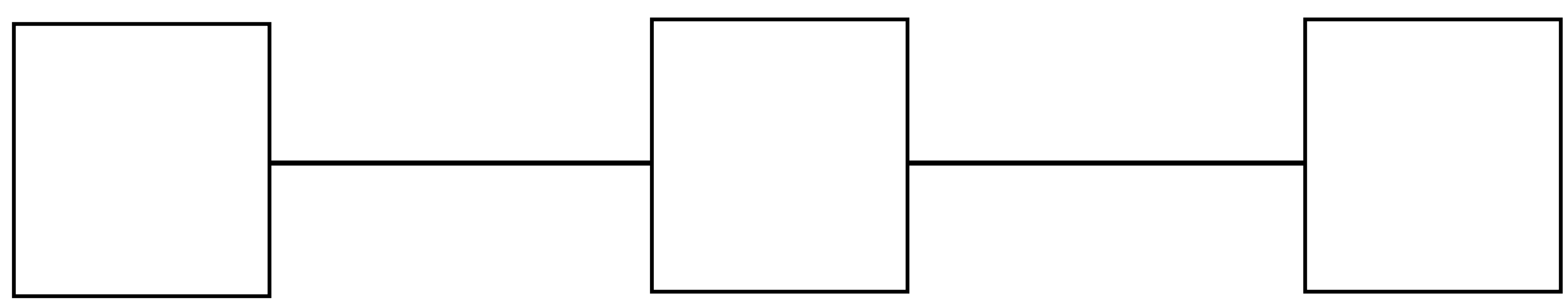
Random codes play an essential role in the theory of error correction

Random codes saturate the channel capacity

C. Shannon, *A mathematical theory of communication*, Bell System Technical Journal 27, 379 (1948).

Practical random LDPC codes with efficient encoders/decoders

R. Gallager, *Low-density parity check codes*, IRE Trans. Info. Theory 8, 21 (1962).



Random Encoding

Local Noise Channel

Optimal Decoding

$$\boldsymbol{x} \to \boldsymbol{y} = f(\boldsymbol{x})$$

Injective

$$P(\boldsymbol{z}|\boldsymbol{y})$$

$$\tilde{\boldsymbol{x}} = \arg\max_{\boldsymbol{x}} P[f(\boldsymbol{x})|\boldsymbol{z}]$$

Random Quantum Coding

Random quantum encodings are Haar random circuits

Decoupling theorem

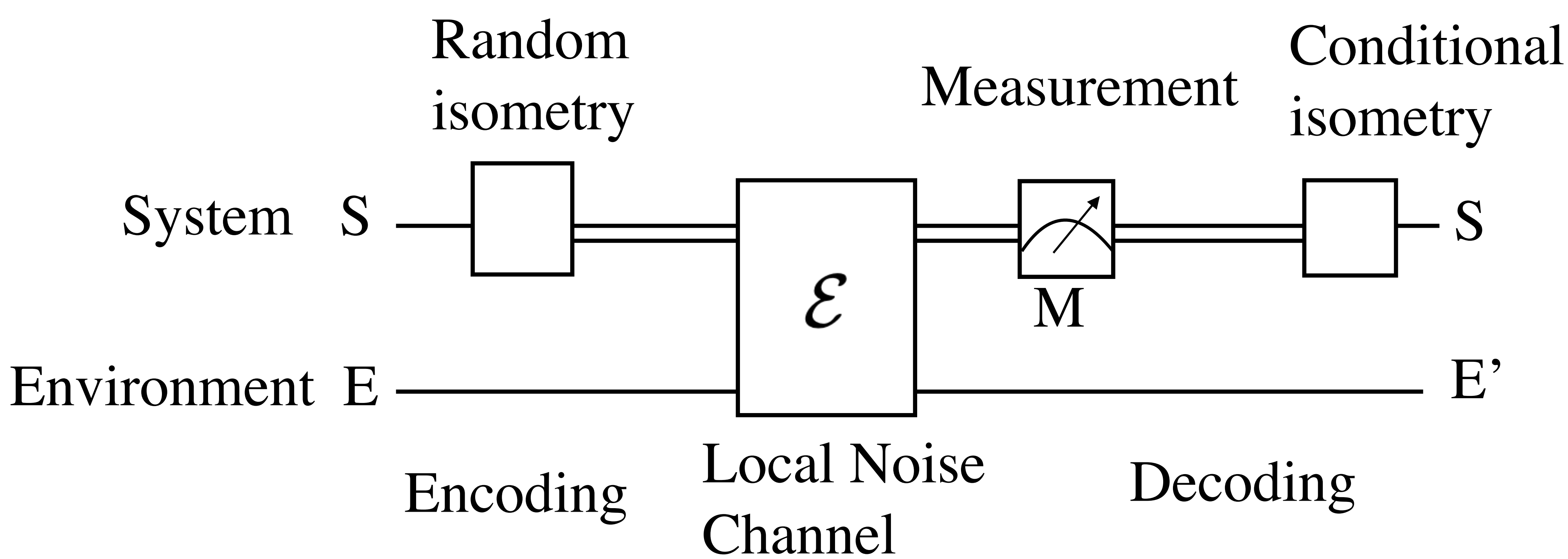
Hayden, Horodecki, Winter, Yard, *A decoupling approach to the quantum capacity,* Open Syst. Inf. Dyn. 15, 7 (2007).

Two-local random Clifford circuits reach linear distance at depth $O(\log^3 N)$

Brown and Fawzi, *Short random circuits define good quantum error correcting codes*, Proceedings of ISIT, pages 346-350 (2013). arXiv:1312.7646

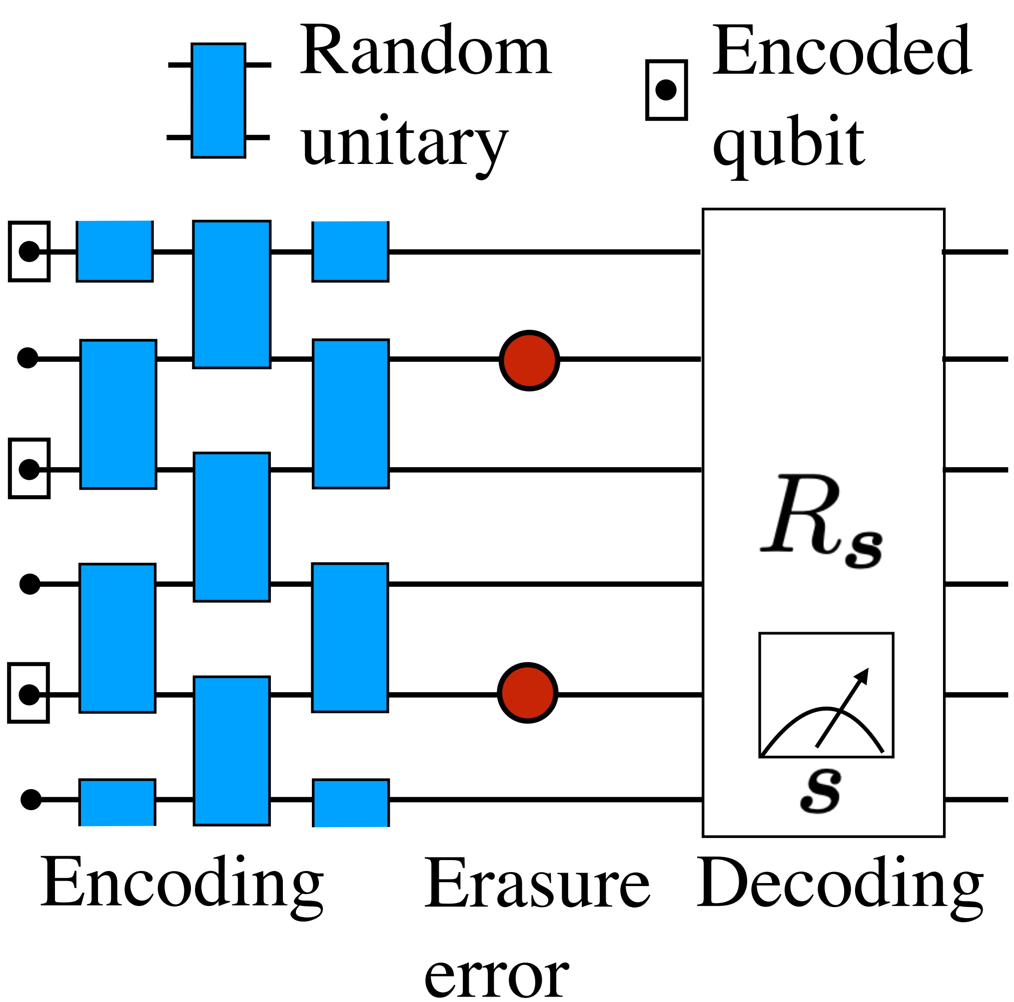
Local random circuits are two-designs at depth $O(N^{1/D})$

Brandao, Harrow, Horodecki, *Local random quantum circuits are approximate polynomial designs*, Commun. Math. Phys. 346, 397 (2016).

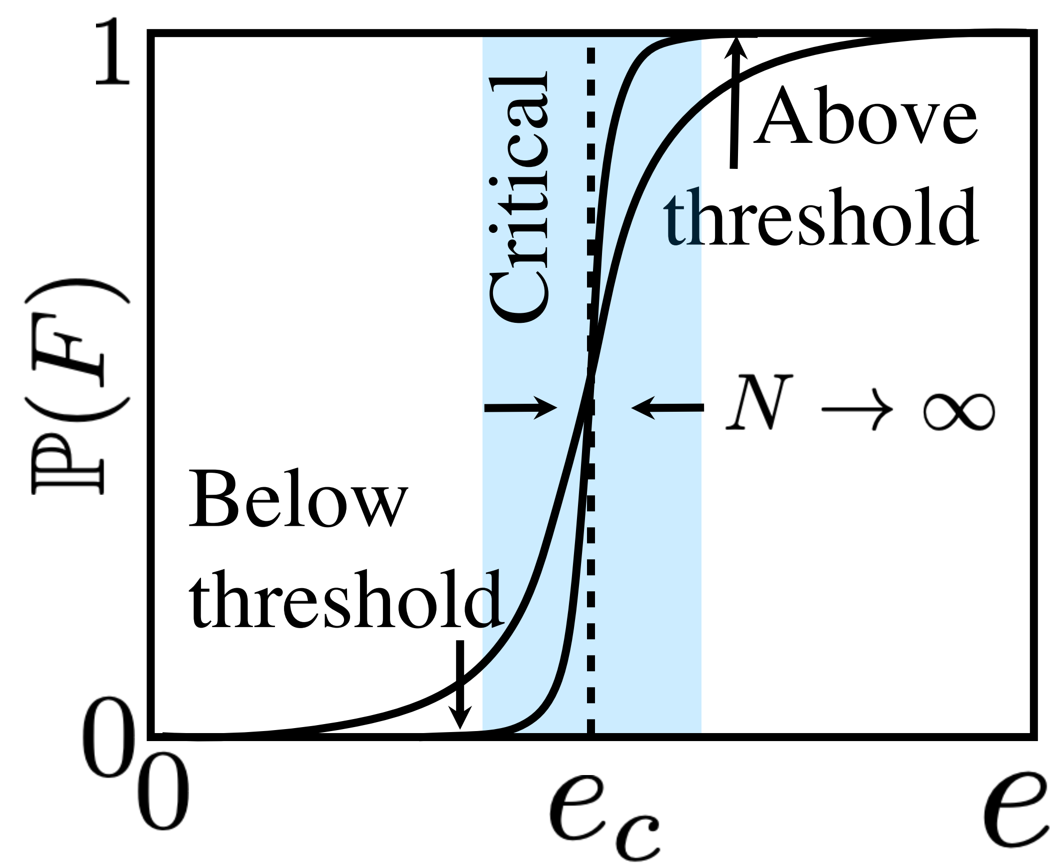


Low-Depth Random Quantum Coding

Study the circuit depth required to converge to zero failure probability for specific model of erasure errors - efficient decoding algorithm*



Erasure - traces out each site with prob. e with heralded locations



Critical erasure rate at capacity:

$$e_c = (1 - R)/2$$

R - code rate

*Delfosse and Zemor, PRR 2, 033042 (2020).

Low-Depth Random Clifford Coding Results

D	$e < e_c$	$e = e_c - O(\frac{1}{N^b})$	Expur. $e < e_c$
1	$(\frac{1}{e_c - e}) \log N$	$N^{1/2}$	$\log N$
2	$\log N$	$\log N$ (conj.)	$(\log N)^c$, $c < 1$
> 2	$\log N$	$\log N$	$(\log N)^c$, $c < 1$

Main results:

1) Coding below capacity requires only O(log *N*) depth in any D

2) Can apply an "Imry-Ma" argument to optimal decoding problem:
Spatial randomness in errors is an irrelevant perturbation for D>2

3) 2D is the marginal dimension for log-*N* depth coding at capacity

4) Introduce an "expurgation" algorithm based on quantum measurements
to achieve sub-log-*N* depth coding below capacity

Random Stabilizer Code Transition - RMT Ansatz

Start with infinite depth limit - Can compute the critical behavior of a random stabilizer code using a random matrix theory ansatz

Fix an erasure pattern, write a generating matrix for all syndromes and logical operator errors

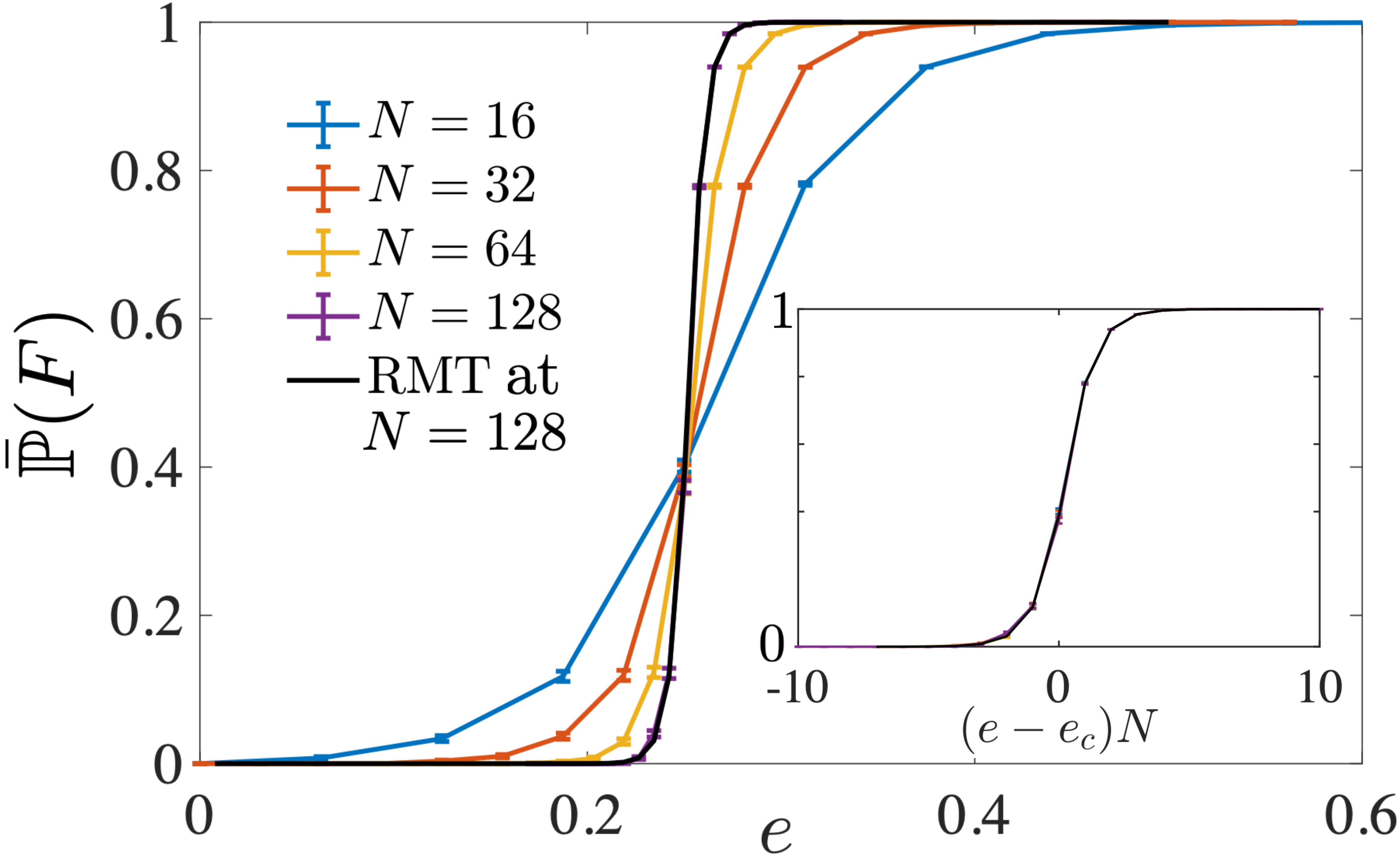
$$M(S, L, \boldsymbol{e}) = \left(\begin{array}{c|c} \boldsymbol{s}_{z_{i_1}} & \boldsymbol{\ell}_{z_{i_1}} \\ \boldsymbol{s}_{x_{i_1}} & \boldsymbol{\ell}_{x_{i_1}} \\ \vdots & \vdots \\ \boldsymbol{s}_{z_{i_{n_e}}} & \boldsymbol{\ell}_{z_{i_{n_e}}} \\ \boldsymbol{s}_{x_{i_{n_e}}} & \boldsymbol{\ell}_{x_{i_{n_e}}} \end{array} \right)$$

In RMT approximation we assume *M* is a random matrix over a finite field

Provides a combinatorial formula for critical recovery probability

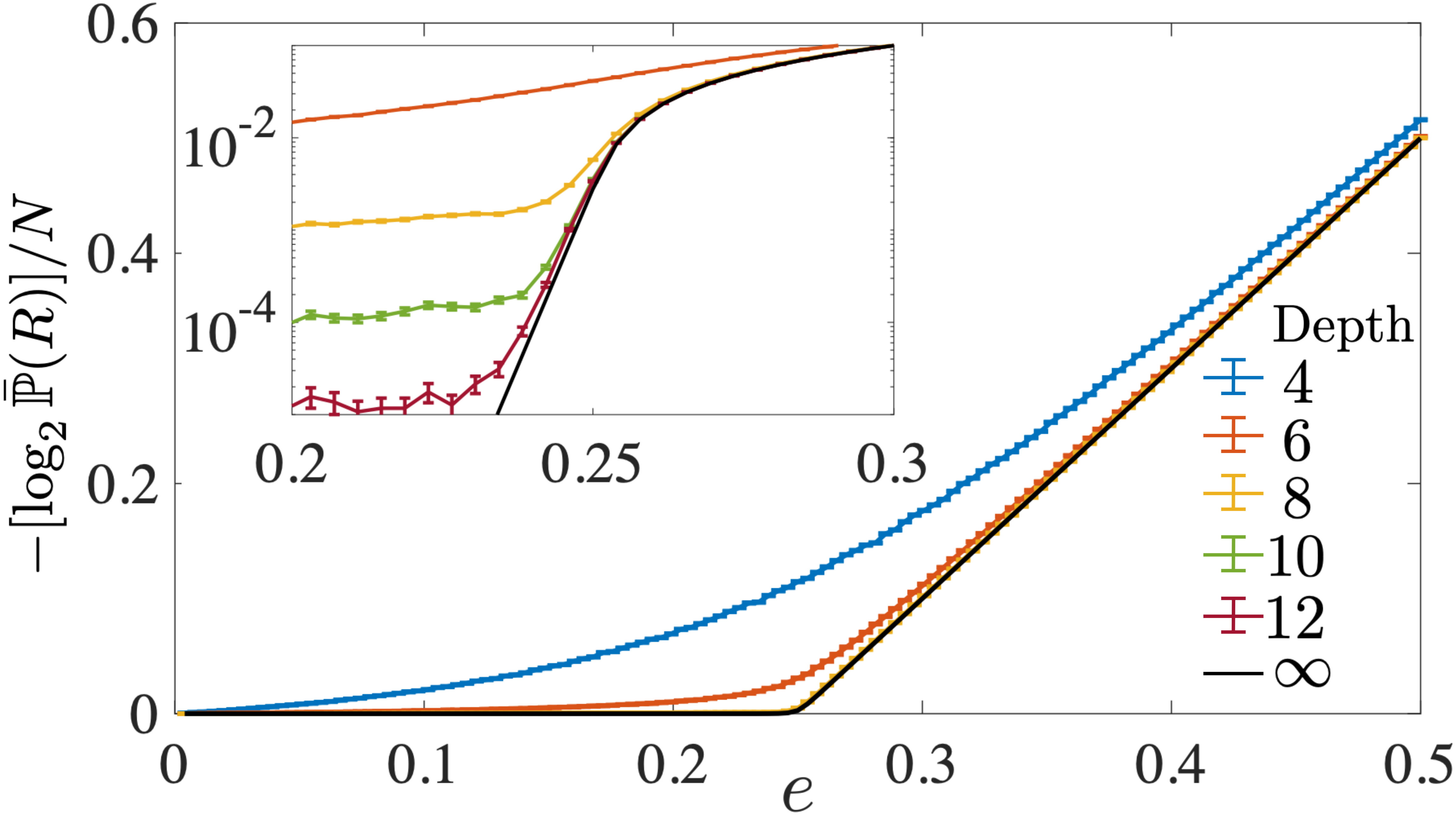
$$\mathbb{P}(R) =_{\mathrm{RMT}} \sum_{m=0}^{\infty} \frac{\prod_{k=1}^{\infty} \left(1 - \frac{1}{2^{m+k}}\right)^2}{2^{m(m+1)} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2^k}\right)} = 0.610322\ldots$$

Random Stabilizer Code Transition - RMT Ansatz



Rate 1/2 code

Low-Depth Random Clifford Codes in 2D

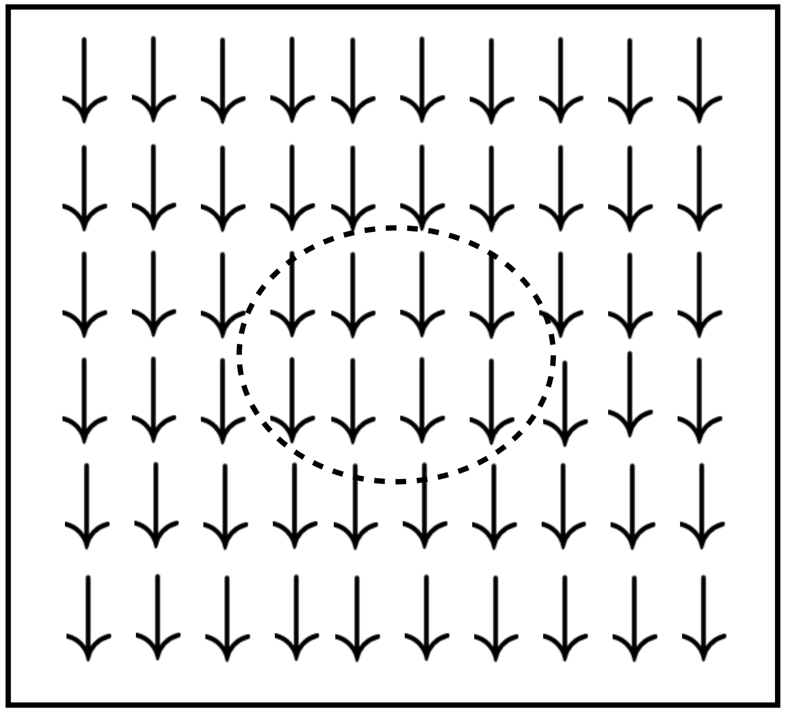


Imry-Ma Argument for Relevance of Randomness

Standard Imry-Ma argument

$$H = -\sum_{\langle ij \rangle} J s_i s_j + \sum_i h_i s_i \quad h_i \in [-h, h]$$

Low-*T*, h = 0



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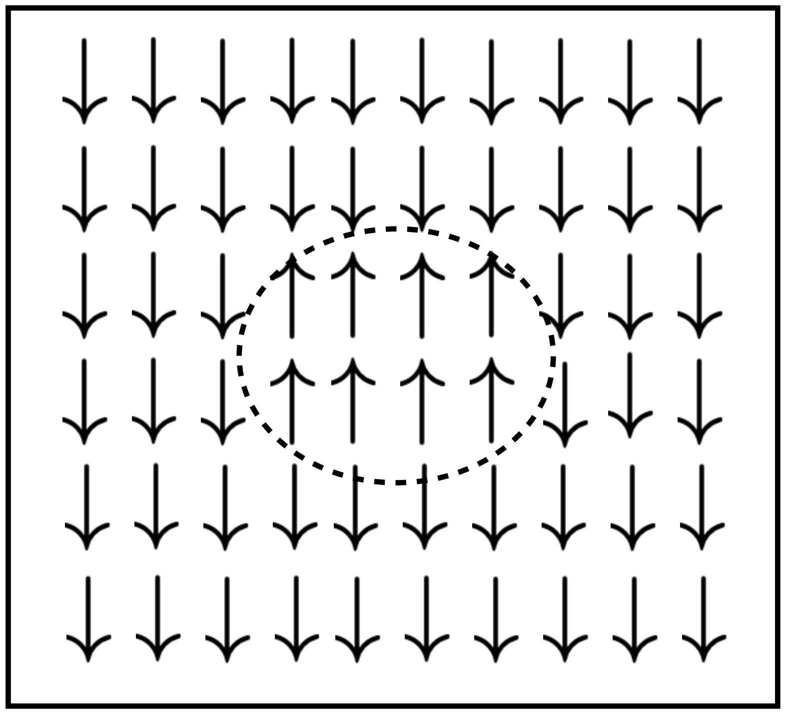


Boundary energy cost -
Ordering tendency

$$\Delta E \sim r^{D-1}$$

Energy gained in field -
Random fluctuations

$$\Delta E \sim h r^{D/2}$$

$$D < 2$$

$$r^{D-1} < r^{D/2}$$

Randomness wins

$$D = 2$$

Marginal dimension
Unresolved*

$$D > 2$$

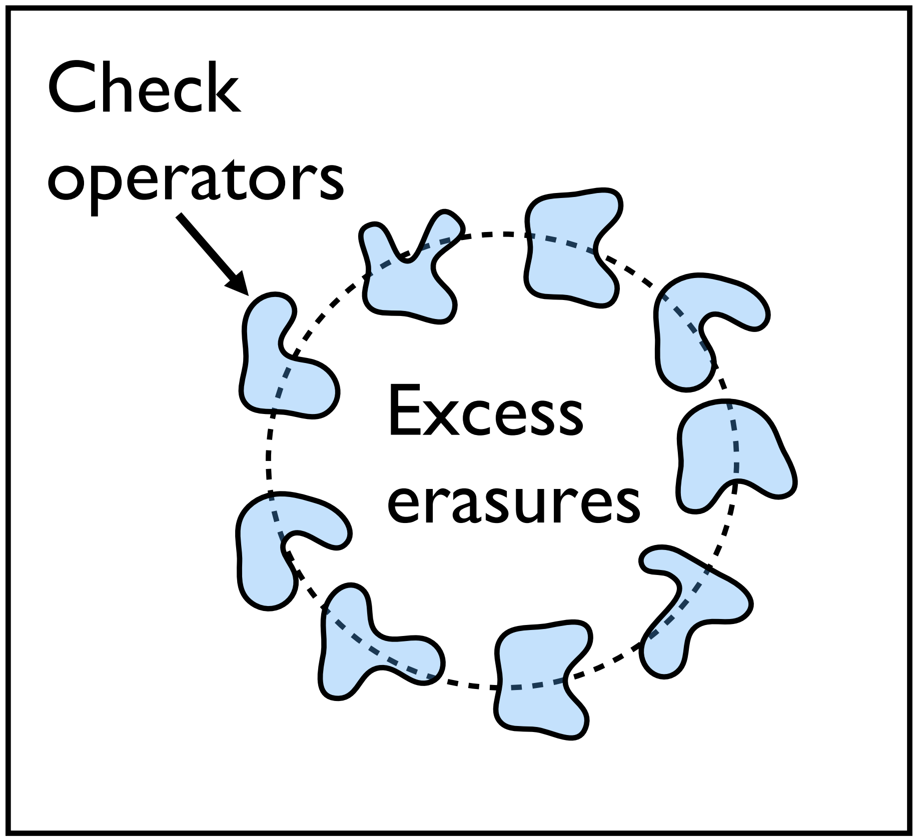
$$r^{D-1} > r^{D/2}$$

Ordering wins

*Recent progress: Aizenmen, Harel, Paled, J. Stat. Phys. 180, 304 (2020).

Imry-Ma Argument for Low-Depth Random Code

Random fields replaced by the random fluctuations in erasure numbers



Number of excess check operators at boundary

$$\sim r^{D-1} d \qquad d \text{ - circuit depth}$$

Random fluctuations in errors

$$\sim r^{D/2}$$

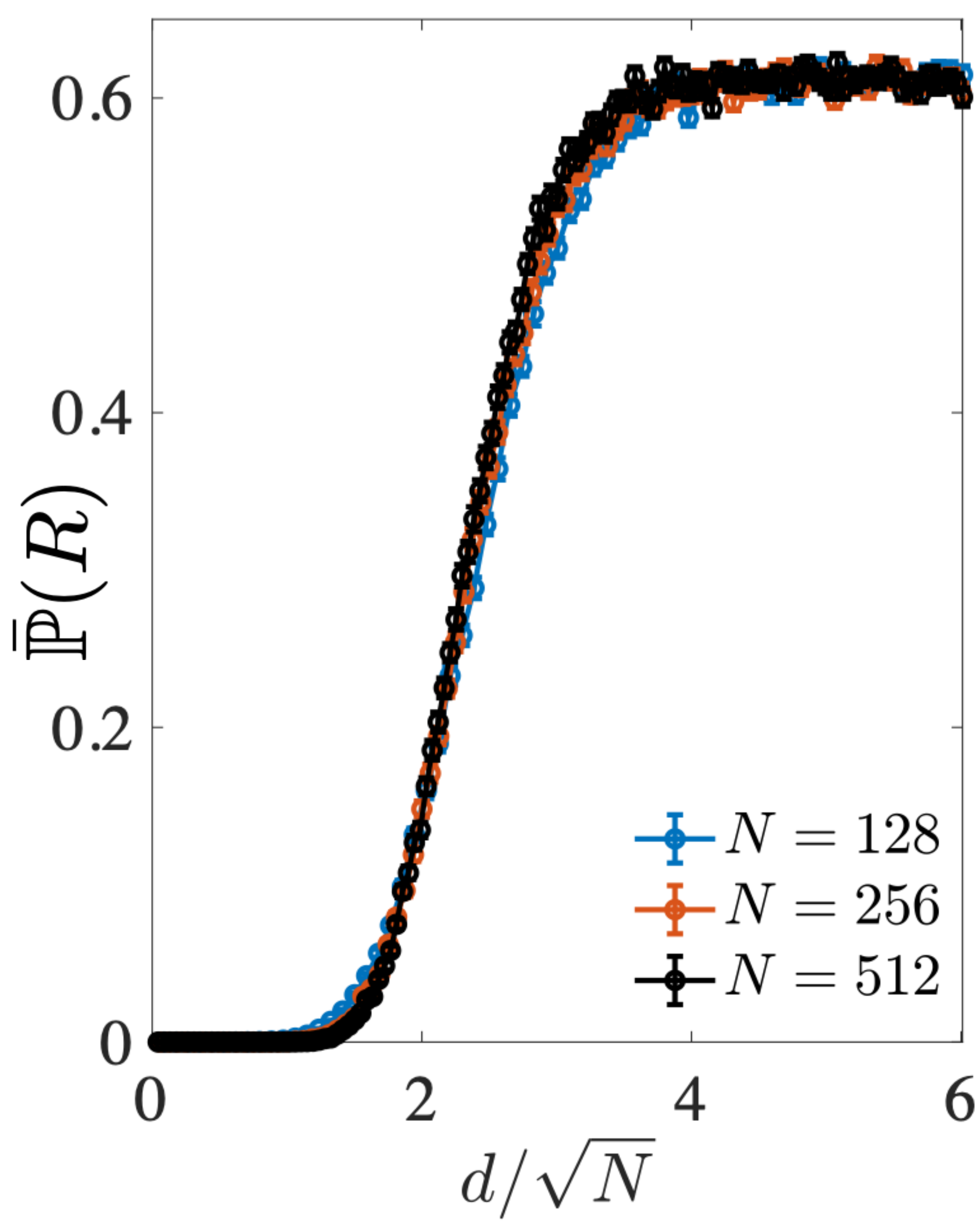
Error correction wins when:

$$d \sim r^{1-D/2}$$

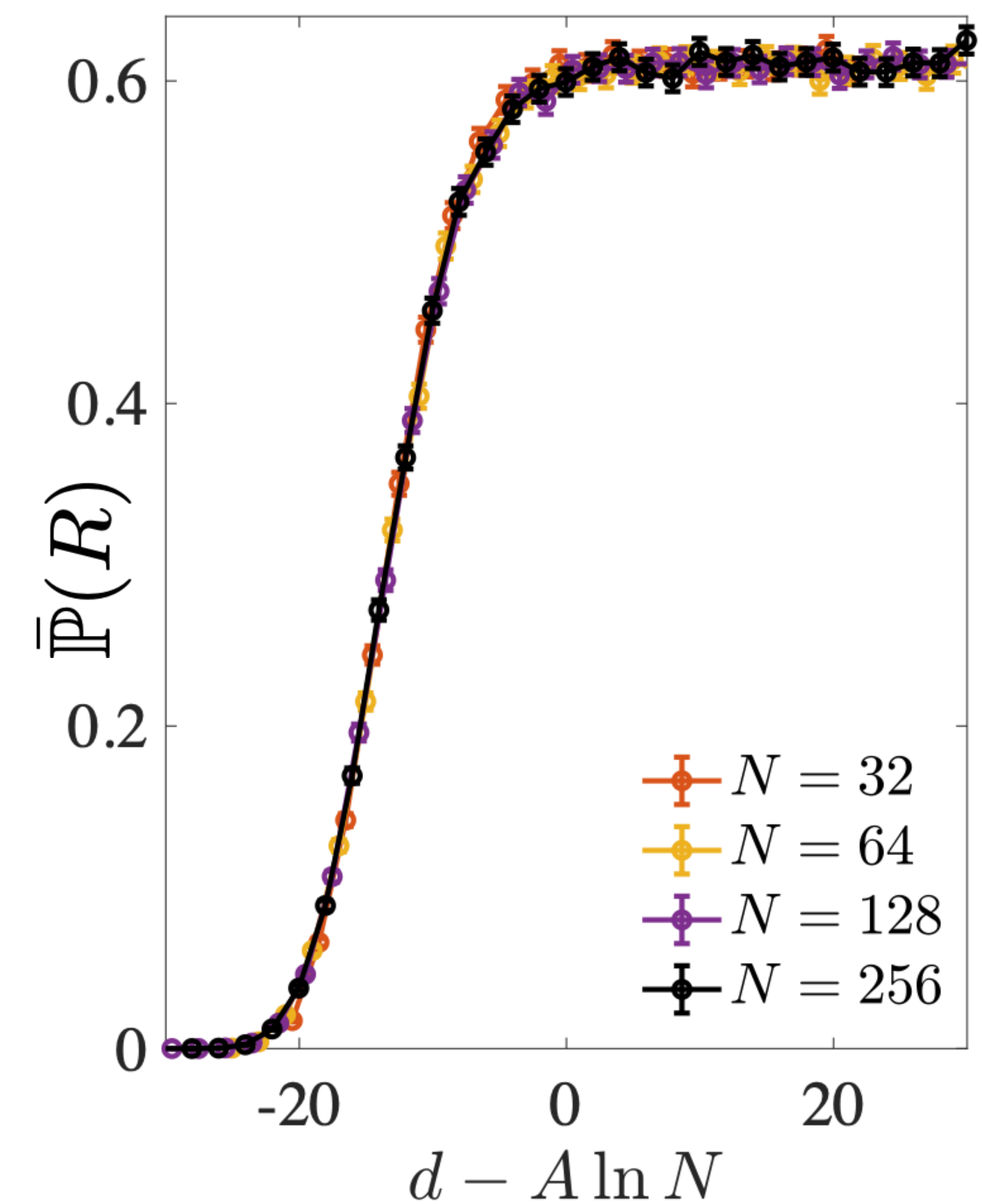
$D < 2$	$D = 2$	$D > 2$
$d = O(\sqrt{r})$		$d = O(1)$
Divergent depth	Marginal dimension Uncertain fate	Error correction wins at constant depth

Random Clifford Coding at Capacity in 1D

Random error locations



Regular error locations



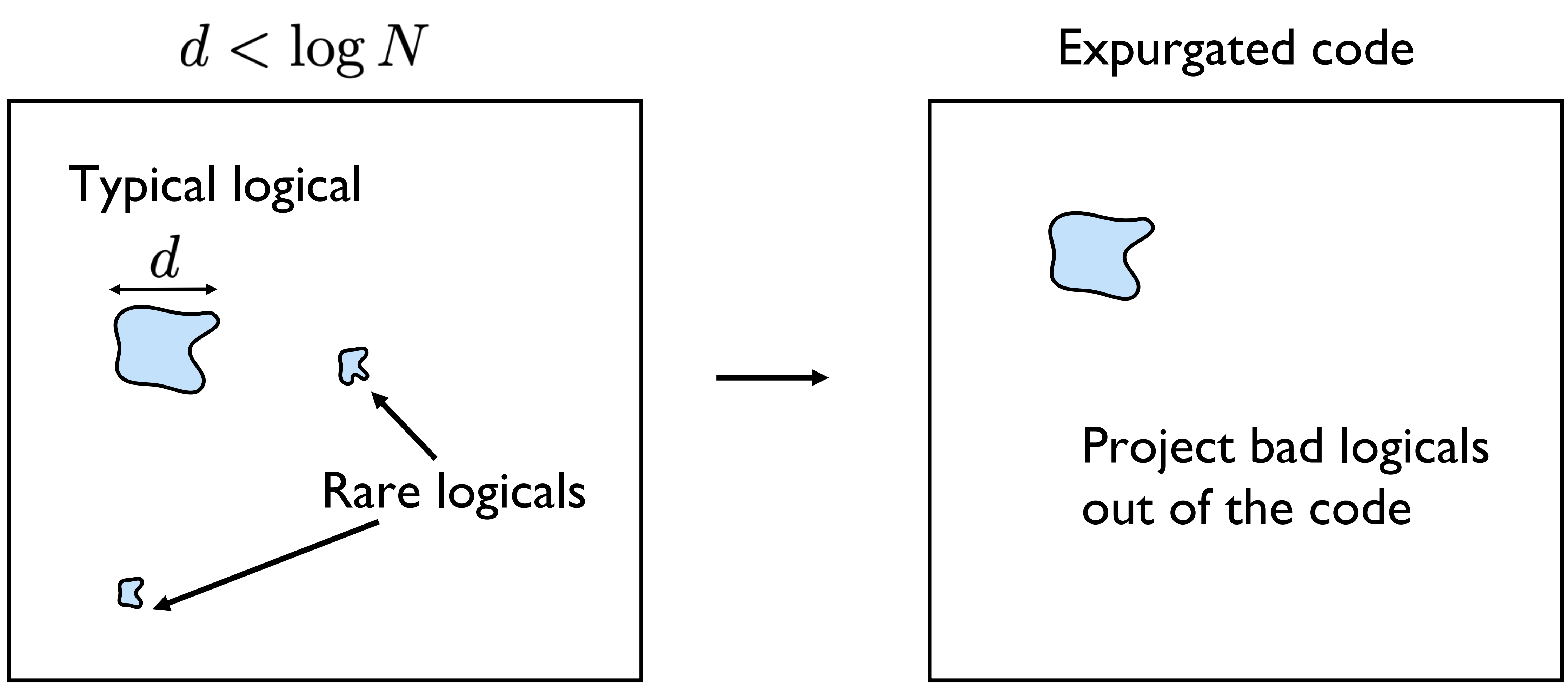
Randomness in the code dominates at log *N* depth

Failure Mode at Low-Depth Below Capacity

Typical minimum weight logical in a given region spreads to a size d^D

Rare localized logicals persist with probability $\sim 1/A^d$ for any D and any random Clifford ensemble

Code fails below log *N* depth for D>1 because of localized logical operators



Expurgation Algorithm Below Capacity

Algorithm for projecting out bad logical operators

Start with a logical operator *g* that has high-probability of leading to failures

Update the code-space density matrix by measuring *g*

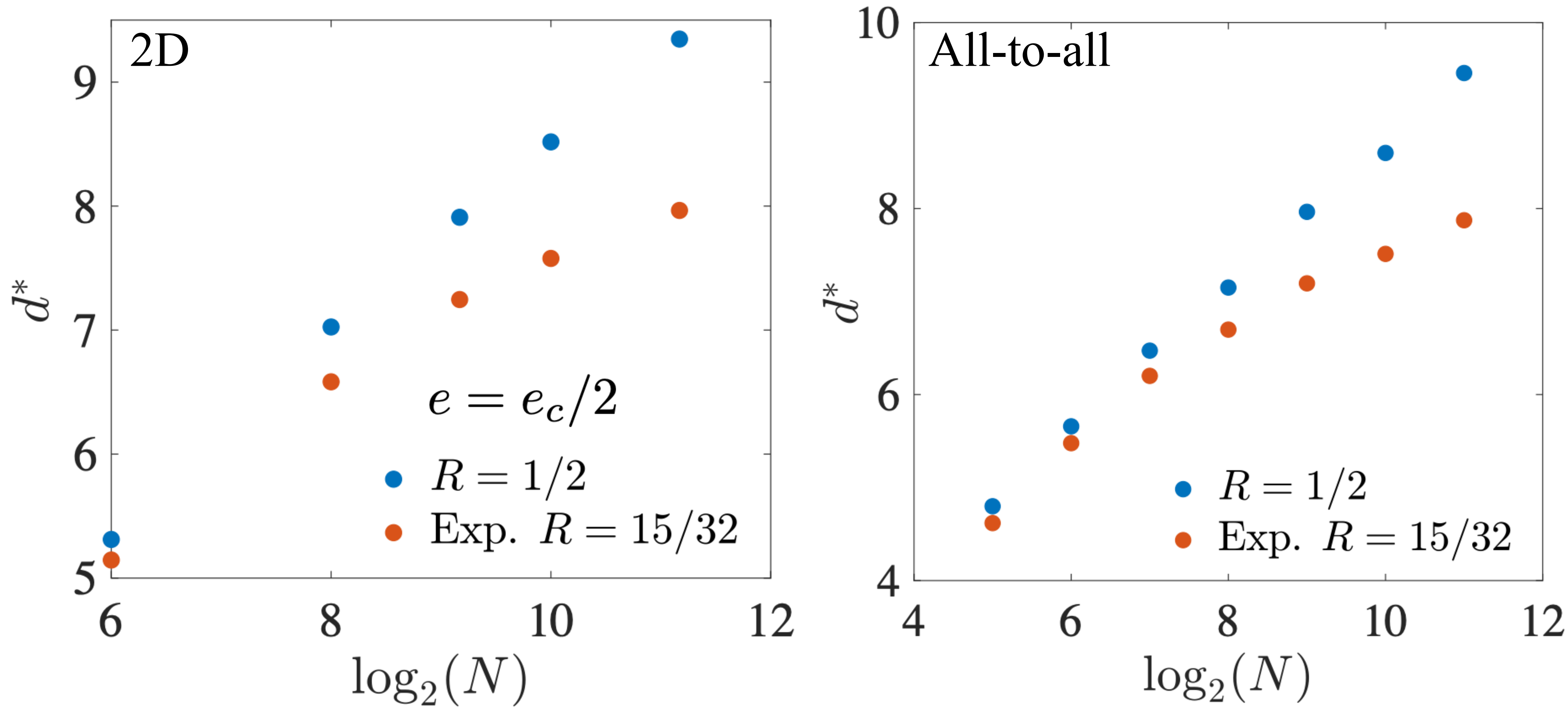
$$\rho_S = \frac{1}{2^N} \prod_{i=1}^{N-k} (\mathbb{I} + \bar{Z}_i) \rightarrow (1 \pm g)\rho_S(1 \pm g)/2$$

$$= \frac{1}{2^N} \prod_{i=1}^{N-k} (\mathbb{I} + \bar{Z}_i)(\mathbb{I} \pm g)$$

Now *g* is a new stabilizer. Find new *g* for the new code. Repeat.

Proposition: This process monotonically increases the code distance and recovery probability without changing original check operators.

Sub-Log-*N* Depth Scaling with Expurgated Random Code



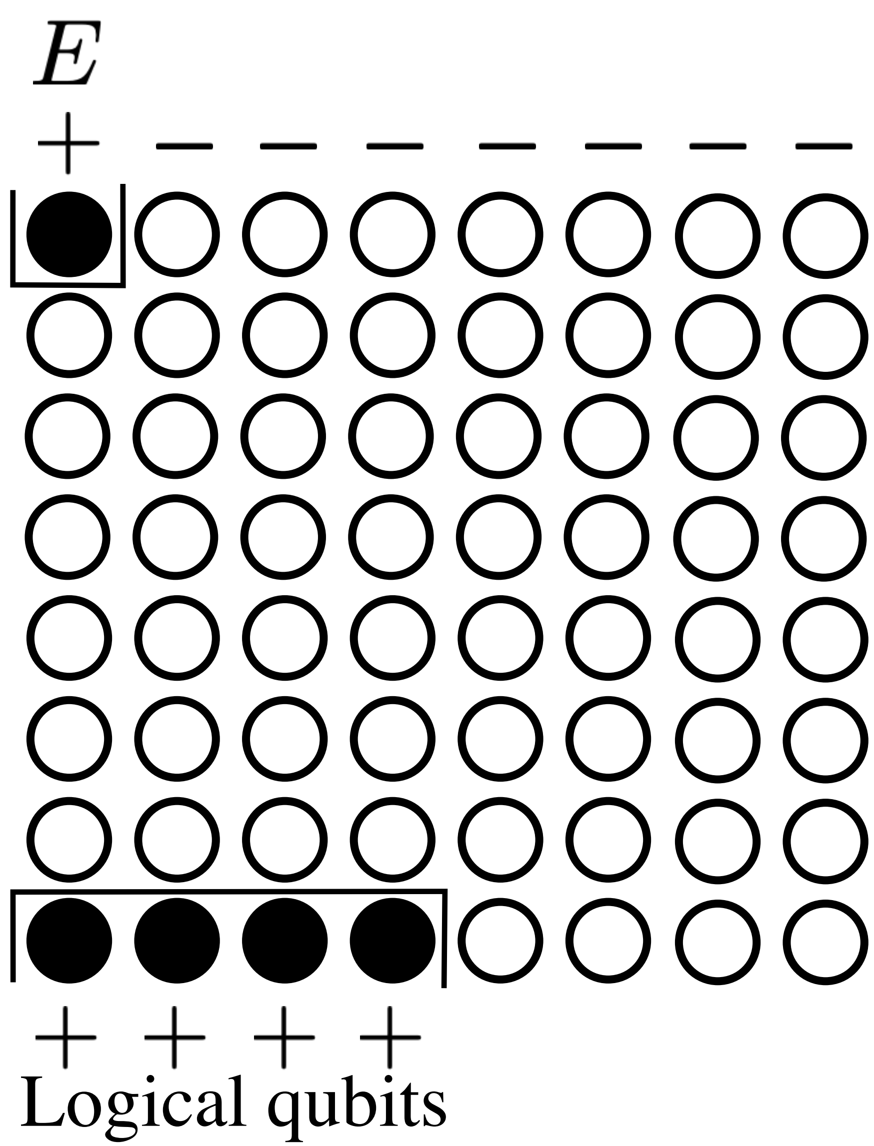
d* is interpolated depth to reach 50% failure probability

No sub-log-*N* scaling observed in *D* = 1

Approximate Stat. Mech. Model for the Erasure Threshold

Use mapping of local random circuit to Ising model for purity of subsystems:
Nahum, Vijay, Haah, PRX 78, 021014 (2018).
Zhou, Nahum, PRB 99,174205 (2019).

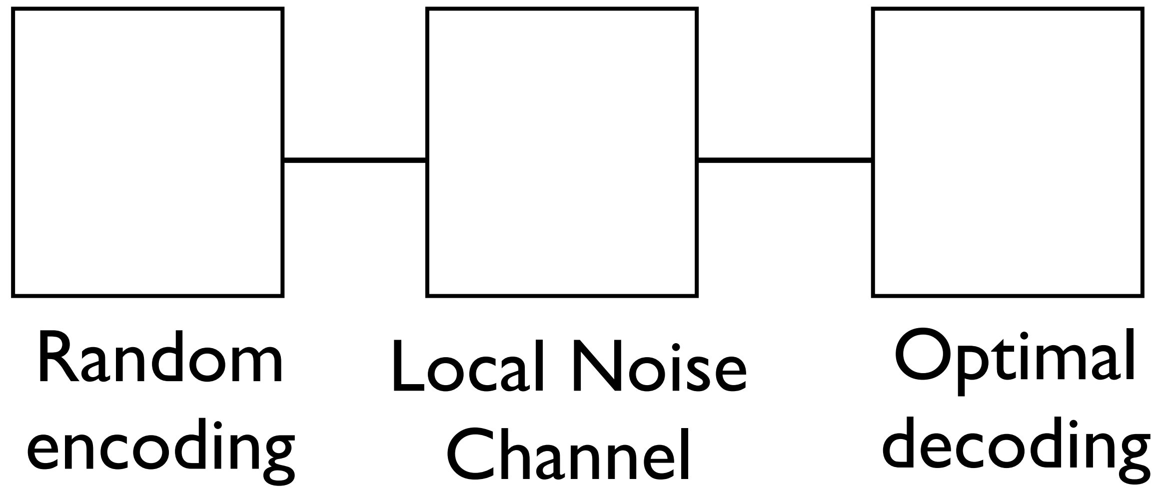
Logical qubits map to polarizing fields on boundary:
Bao, Choi, Altman, PRB 101, 104301 (2020).



Erasure threshold is a first-order pinning transition for this domain wall

Summary

Random channel coding



1) Very high-performance codes exist at very low depths!

2) Deep connections between random quantum coding and statistical physics

3) Expurgation algorithm points to the untapped potential of spatially local finite-rate codes

Reference:
arXiv:2010.09775

Open Questions

- Optimal decoding for low-depth codes beyond erasure errors

- Performance with circuit-level errors - towards fault-tolerance

- Biased random codes: Can we develop biased sampling techniques like expurgation to find noise-optimized random codes?

- Adaptive decoding: Are there efficient adaptive strategies that dynamically update the code space depending on observed syndromes?

- Connections to measurement-induced entanglement phase transitions

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