Efficient unitary designs with a system-size independent number of non-Clifford gates

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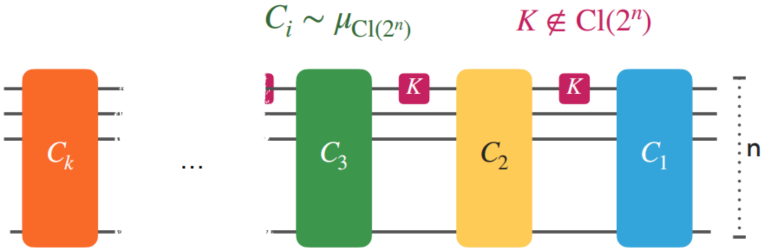
arXiv:2002.09524

"Quantum homeopathy": the result in a nutshell



is an approximate unitary *t*-design in depth $k \geq O(t^4)$ provided that $n \geq O(t^2)$.

Random unitaries are everywhere

- Quantum system identification: randomized benchmarking, tomography, shadow tomography...
- Sending information through quantum channels.
- Models of quantum information scrambling for black holes.
- Generic features of quantum many-body systems.

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But: full Haar-randomness requires exponential resources!

Unitary designs

Definition

Consider the moment operator:

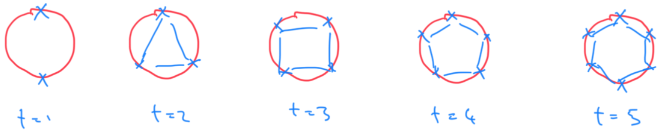
$$\Delta_{\nu,t}(A) := \mathbb{E}_{U\sim\nu} U^{\otimes t} A (U^{\dagger})^{\otimes t} \tag{1}$$

Exact unitary t-design is a probability measure ν such that

$$\Delta_{\nu,t} = \Delta_{\mu_{\mathrm{Haar}},t}. \tag{2}$$

Unitary designs

Measures how evenly spread a set of unitaries is. E.g. the time evolution under random Hamiltonians.



Discrete designs always exist but are complicated!

Unitary group designs

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Well behaved case: ν is Haar measure on subgroup G of $U(N)$. Then $\Delta_{\nu,t}$ is orthogonal projector on $\mathrm{Comm}(U^{\otimes t}, U \in G)$.

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Example: $\Delta_{\mu_{\mathrm{Haar}},t}$ is projector on

$$\mathrm{Comm}(U^{\otimes t}, U \in U(N)) = \mathrm{span}(r(\pi), \pi \in S_t) \qquad \text{(Schur-Weyl)}$$

The Clifford group: many faces

- Generated by Phase gate, Hadamard and controlled NOT: $\mathrm{Cl}(2^n) := \langle\{S, H, \mathrm{CZ}\}\rangle$.
- Normalizer of the Pauli group: $\mathrm{Cl}(2^n) := \{U \in U(2^n), U\mathcal{P}U^\dagger \subseteq \mathcal{P}\}$.
- Symplectic group on vector space over finite field.
- Analogue of Gaussian operations for discrete variables.

The Clifford group: many applications

- Quantum error correction.
- Randomized benchmarking.
- Simulation of quantum circuits.
- (Shadow) tomography.
- ...

The Clifford group as a unitary design

PHYSICAL REVIEW A **96**, 062336 (2017)

Multiqubit Clifford groups are unitary 3-designs

Huangjun Zhu*

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ZAK WEBB[1]

Quantum Information & Computation, 2016

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Another face: Multiqubit Clifford group is unique non-trivial 3-group in $U(2^n)$.

Bannai, Navarro, Rizo, Tiep, arXiv:1810:02507 (2018)
Guralnick, Tiep, Representation Theory (2005)
Sawicki, Karnas, Ann. Henri Poincaré (2017).

No good 4-designs?

$\implies$ 4-groups don't exist!

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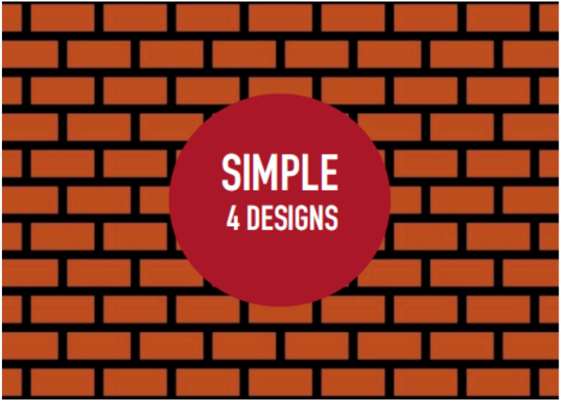

Unitary designs via random quantum circuits

Definition

ν is ε-approximate design if

$$||\Delta_{\mu_{\text{Haar}},t} - \Delta_{\nu,t}||_\diamond \leq \varepsilon. \quad (3)$$

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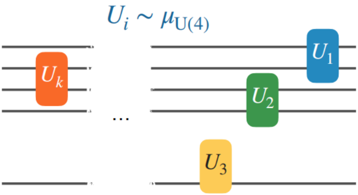
Theorem (Brandão, Harrow, Horodecki)

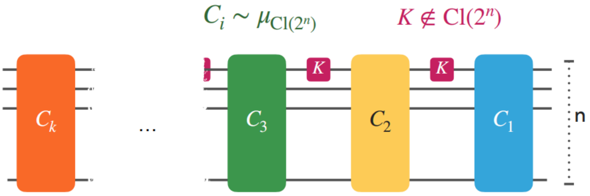


is an approximate unitary t-design for $k \geq O(n^2 t^{9.5})$.

Brandão, Harrow, Horodecki, Comm. in Math. Phys. (2016).

Unitary t-designs from random Clifford dominated circuits

Theorem

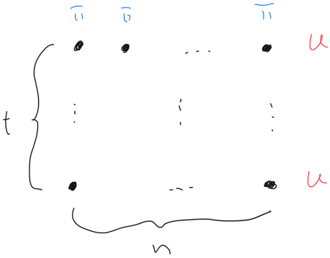


is ε-approximate unitary design in depth $k \geq C_1(K)(t^4 + \log(1/\varepsilon))$ *for all* $n \geq C_2(K)t^2$.

Schur-Weyl duality (for the Clifford group)

For $\pi \in S_t$, we denote

$$r(\pi) := 2^{-t/2} \sum_{x \in \mathbb{Z}_2^t} |\pi(x)\rangle\langle x|.$$



$$\mathrm{Comm}(U^{\otimes t}) = \mathrm{span}(r(\pi)^{\otimes n})$$

$$\mathrm{Comm}(U^{\otimes t}, U \in \mathrm{Cl}(2^n)) = \mathrm{span}(r(T)^{\otimes n})$$

$T \in \Sigma_{t,t}$ Lagrangian subspaces.

Gross, Nezami, Walter, arXiv:1712:08628, (2017)

Proof strategy

Rewrite moment operator

$$\Delta_{\sigma,t} - \Delta_{\mu_{\mathrm{Haar}},t} = [(\underbrace{\Delta_{\mu_{\mathrm{Cl},t}} - \Delta_{\mu_{\mathrm{Haar}}}}_{\text{projector}})\mathrm{Ad}_{K^{\otimes t}}]^k. \quad (4)$$

- $\Delta_{\mu_{\mathrm{Cl},t}} - \Delta_{\mu_{\mathrm{Haar}}}$ is projector onto the orthocomplement of the permutations in the span of Lagrangian subspaces.
- Technical problem: $\{r(T)^{\otimes n}\}$ is not an orthonormal basis.
- Careful bound on Gram-Schmidt orthogonalization of Lagrangian subspaces for $n \geq O(t^2)$.

Proof strategy

- Painful combinatorial argument yields

$$||\Delta_{\sigma,t} - \Delta_{\mu_{\mathrm{Haar}},t}||_\diamond \leq 2^{O(t^4)+t\log(k)}(1+2^{O(t^2)-n})^{5k}\eta^{k-1}.$$

- Depends on "expectation value" of $\mathrm{Ad}_{K^{\otimes t}}$ acting on non-permutation Lagrangian subspaces:

$$\eta := \max_{T\in\Sigma_t\setminus S_t} \left|\left(r(T)|\mathrm{Ad}_{K^{\otimes t}}|r(T)\right)_{\mathrm{HS}}\right|.$$

Lemma

$$\eta \leq 1 - c(K)\log^{-2}(t).$$

Open problems

- Can the condition $n \geq O(t^2)$ be lifted?
- Relative approximate designs. $\rightarrow$ partial result: $O(n)$ instead of $O(n^2)$ many non-Clifford gates.
- Applications of higher designs:

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- Can the condition $n \geq O(t^2)$ be lifted?
- Relative approximate designs. $\rightarrow$ partial result: $O(n)$ instead of $O(n^2)$ many non-Clifford gates.
- Applications of higher designs:
  - Out-of-time-order correlators and scrambling?
  - Quantum PUFs?
  - Equilibration.
  - Complexity growth.
  - ...

Mi, et. al., arXiv:2101.08870
Kumar, Mezher, Kashefi, arxiv:2101.05692