Quantum speedups for graph sparsification, graph cut problems and Laplacian solving

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(arXiv:1911.07306, arXiv:2011.09823)

Graph model

Graph model

undirected, weighted graph $G = (V, E, w)$

n nodes and m edges

$\in O(n^2)$

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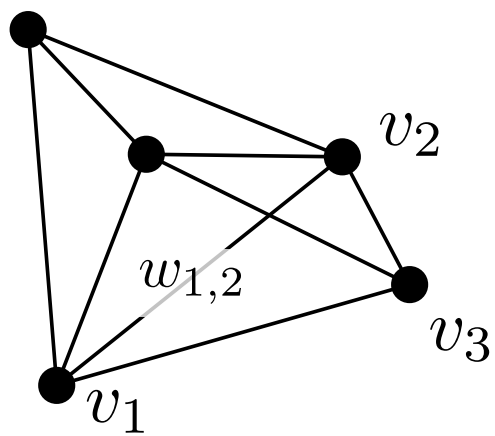
adjacency list access:

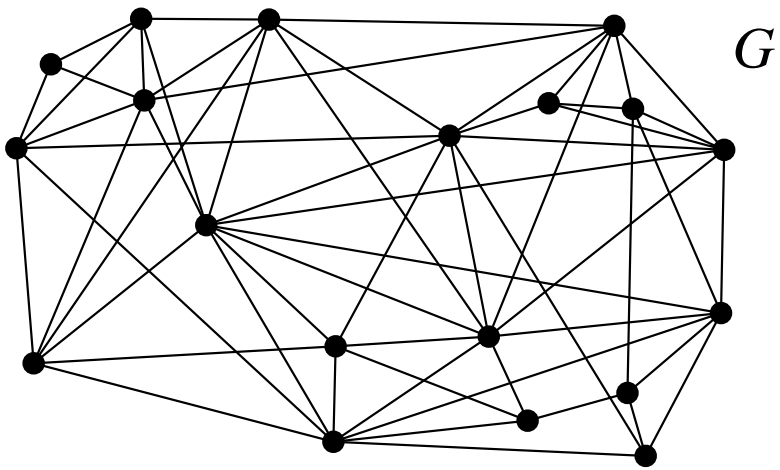
(v_1): $(v_2, w_{1,2})$, $(v_3, w_{1,3})$, ...
(v_2): $(v_3, w_{2,3})$, ...
⋮
(v_n): ...

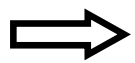


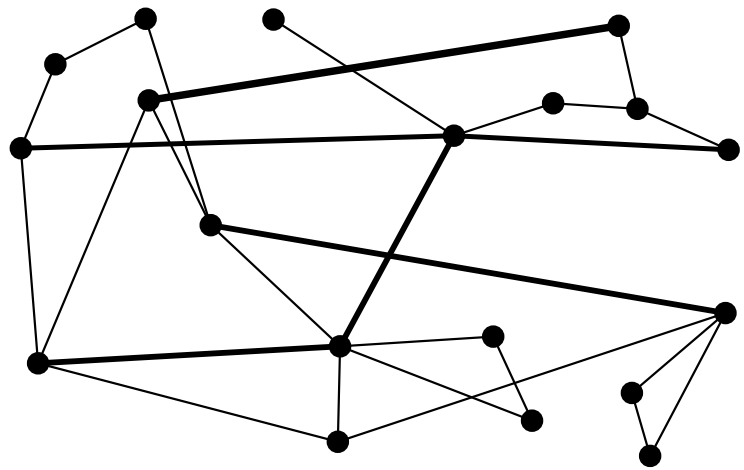
Graph Sparsification

"graph sparsification"

= reduce number of edges, while preserving interesting quantities







$m_G \in O(n^2)$

$m_H << m_G$

Graph Sparsification

"**spectral** sparsification"

Graph Sparsification

"**spectral** sparsification"

= approximately preserve all quadratic forms:

$$x^T L_H x = (1 \pm \varepsilon)\, x^T L_G x, \quad \forall x \in \mathbb{R}^n$$

this includes:

- cut values
- eigenvalues
- random walk properties
- ...

Graph Sparsification

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Karger '94, Benczúr-Karger '96, Spielman-Teng '04, Batson-Spielman-Srivastava '08:

Theorem

Every graph has ϵ-spectral sparsifier H with a number of edges

$$\widetilde{O}(n/\epsilon^2) \quad << O(n^2)$$

which can be found in time $\widetilde{O}(m)$.

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- building block of many $\widetilde{O}(m)$-time **approximation** algorithms
- crucial component of Spielman-Teng $\widetilde{O}(m)$-time algorithm for solving **Laplacian system** $Lx = b$

This work:

$m \geq \sqrt{mn}/\varepsilon \geq n/\varepsilon^2$, $\varepsilon \leq \sqrt{\frac{m}{n}}$

This work:

1. **quantum algorithm** to find ϵ-spectral sparsifier H in time

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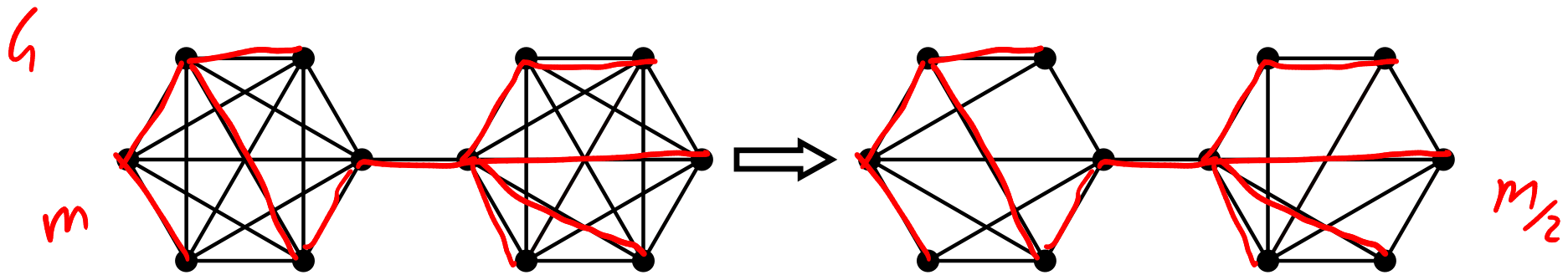
2. **quantum algorithm** for finding *exact* minimum cut

Classical Sparsification Algorithm

Iterative sparsification [Koutis-Xu '16]:

1. construct $\widetilde{O}(1/\epsilon^2)$ "graph spanners" and keep these edges
2. keep any remaining edge with probability $1/2$, and double its weight



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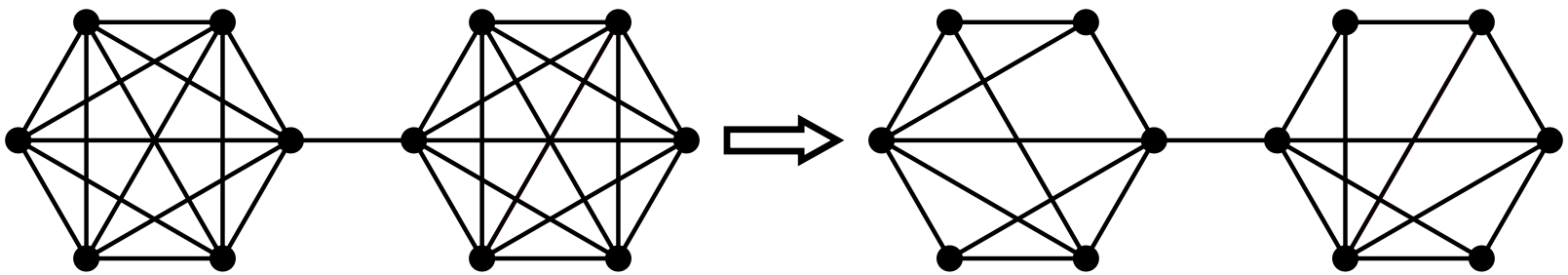
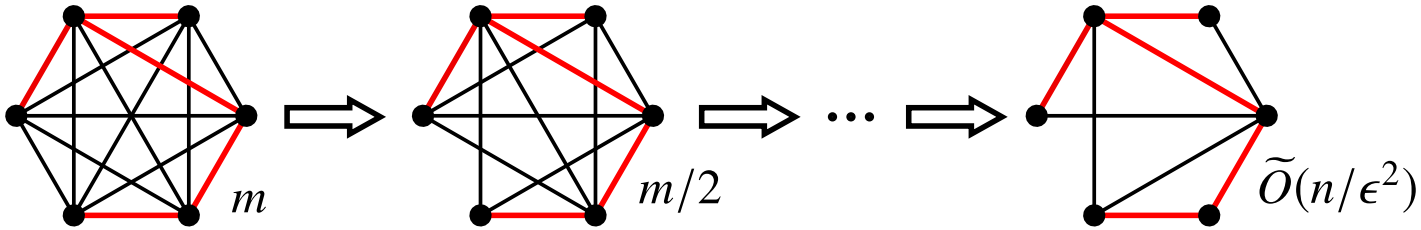


$\rightarrow$ *repeat* $O(\log n)$ *times:* ϵ-spectral sparsifier with $\widetilde{O}(n/\epsilon^2)$ edges

Quantum Sparsification Algorithm



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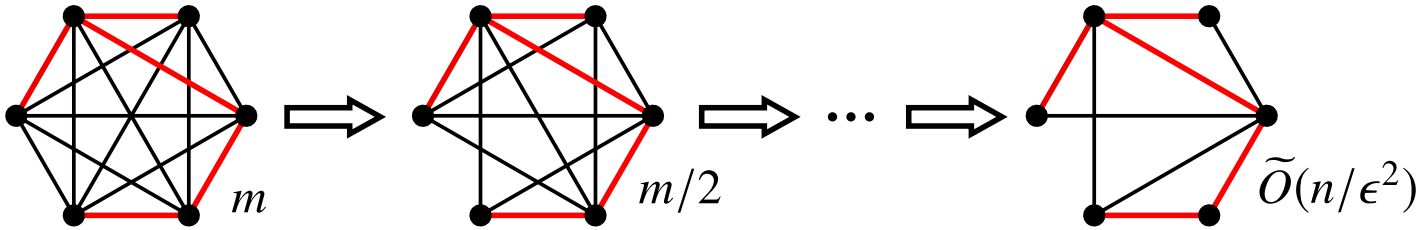

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complexity $\widetilde{O}(\sqrt{mn})$

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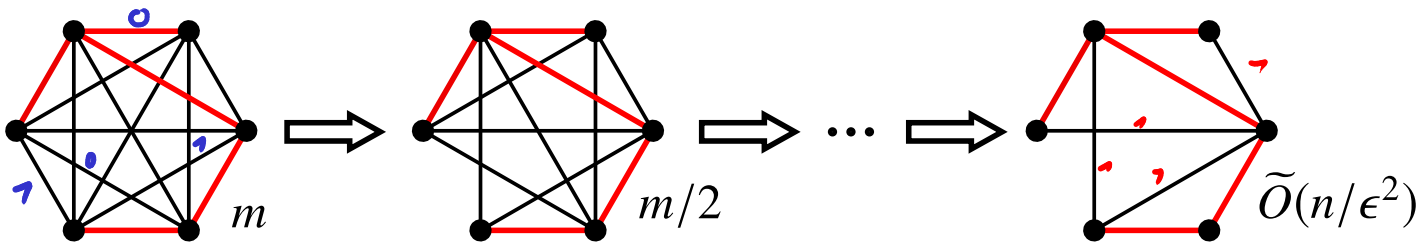

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+ k**-independent oracle**

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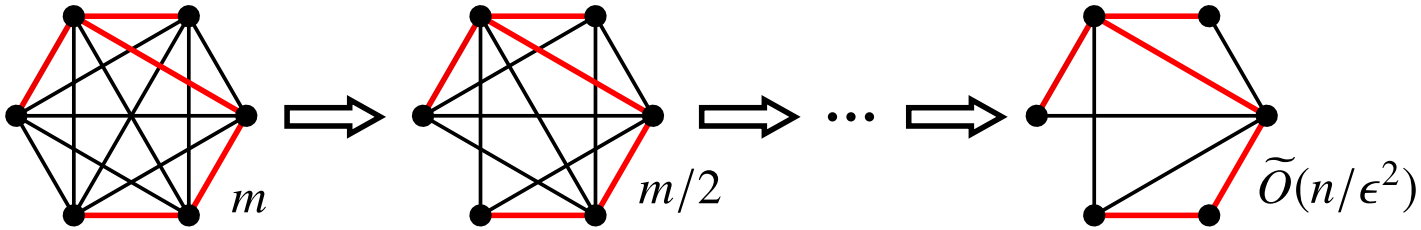

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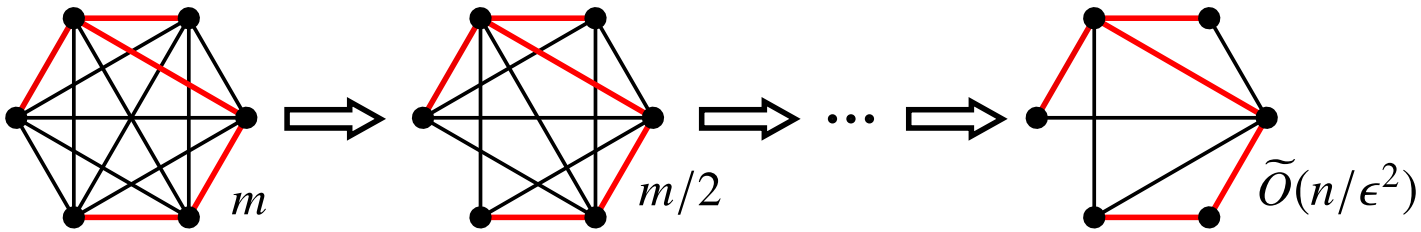

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improve ϵ-dependency

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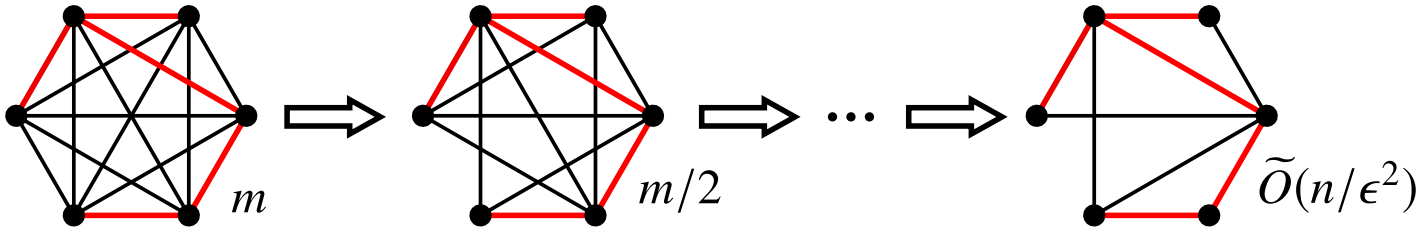

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Quantum Sparsification Algorithm

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There is a quantum algorithm that constructs an ϵ-spectral sparsifier with $\widetilde{O}(n/\epsilon^2)$ edges in time

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This is tight (up to polylog-factors).

Matching Quantum Lower Bound

intuition:

finding k marked elements among M elements takes

$\Omega(\sqrt{Mk})$ **quantum queries**,

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where “hence”

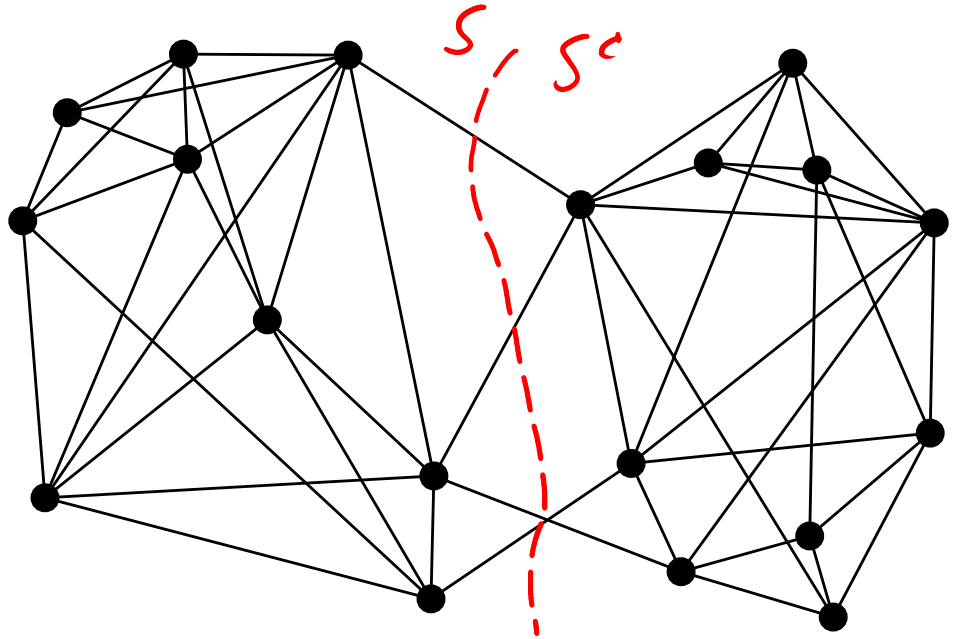
=

hiding an unsparsifiable graph [Andoni-Chen-Krauthgamer-Qin-Woodruff-Zhang ’16]

+ quantum lower bound for relational problem [Belov-Lee ’20]

Applications

e.g., MIN CUT:



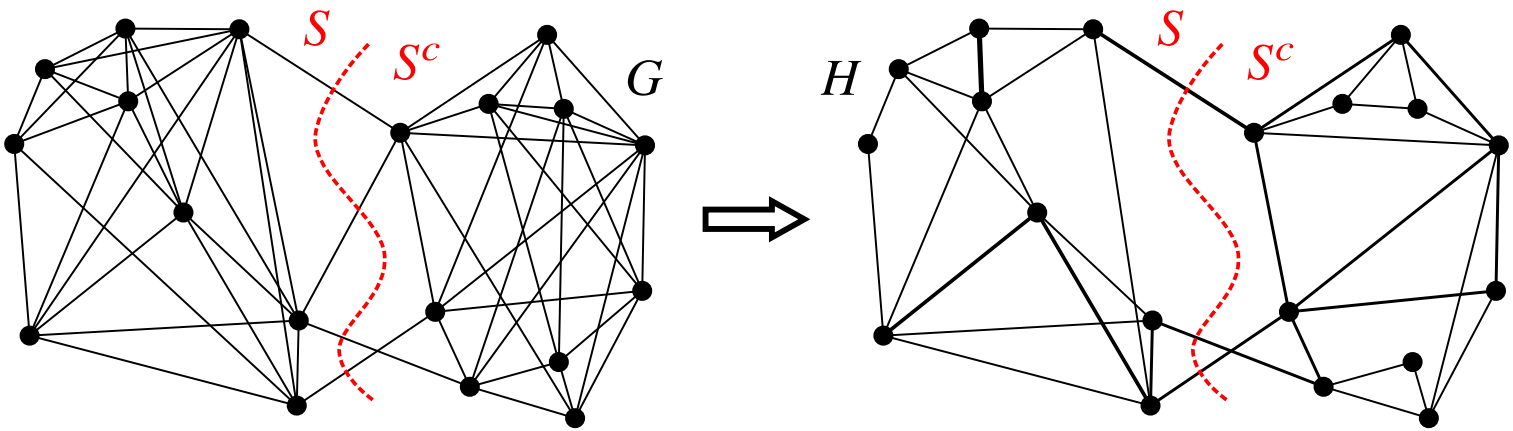
find cut (S, S^c) that minimizes cut value

$$\text{cut}_G(S) = \sum_{i \in S, j \notin S} w_{i,j} \quad = 4$$

Applications

MIN CUT of ϵ-spectral sparsifier H
gives ϵ-approximation of MIN CUT of G

$O(n^2)$



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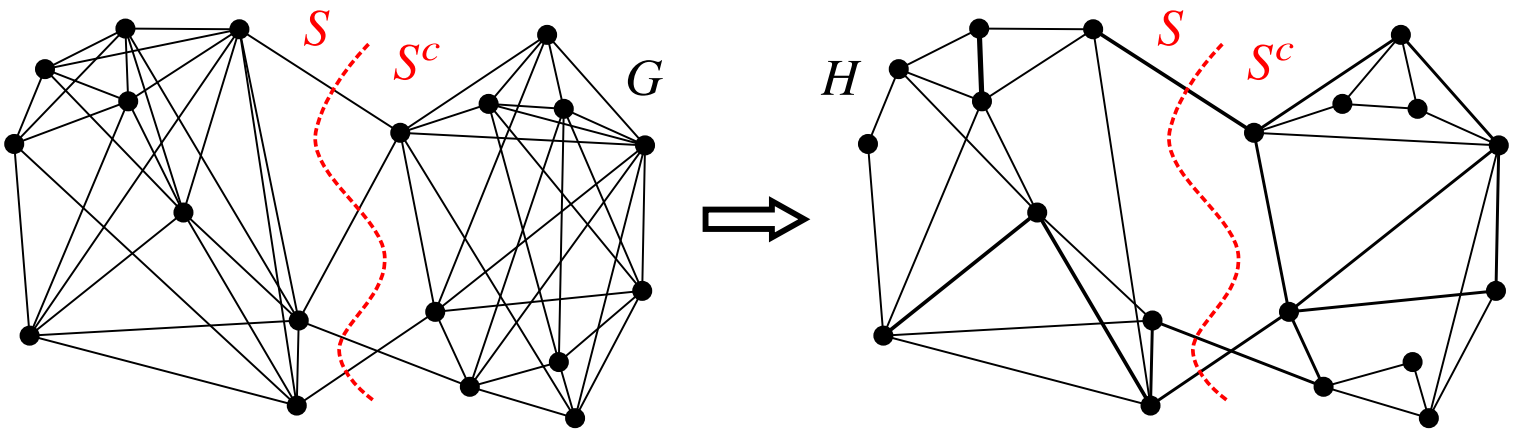

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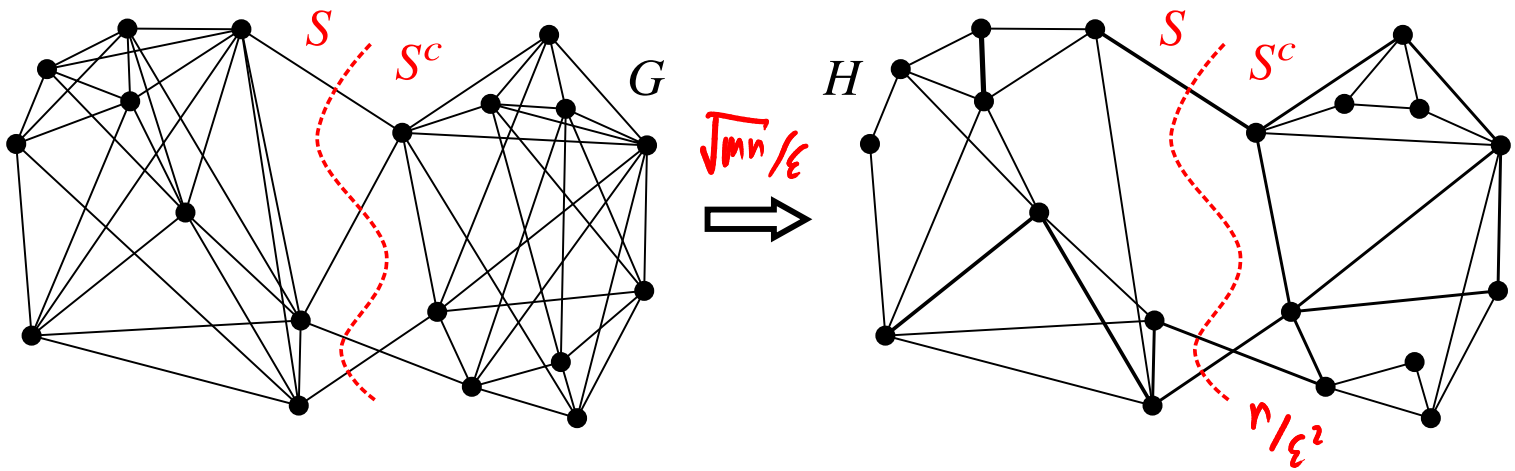

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quantum algorithm to create sparsifier H in $\widetilde{O}(\sqrt{mn}/\epsilon)$
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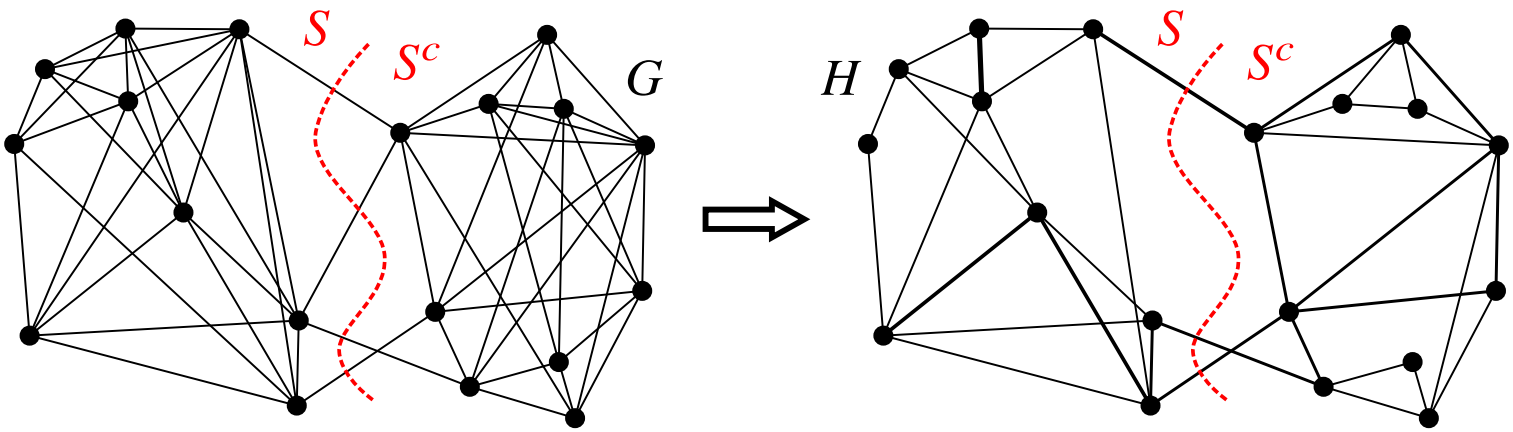

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= $\widetilde{O}(\sqrt{mn}/\epsilon)$ quantum algorithm

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$\rightarrow$ general blueprint:

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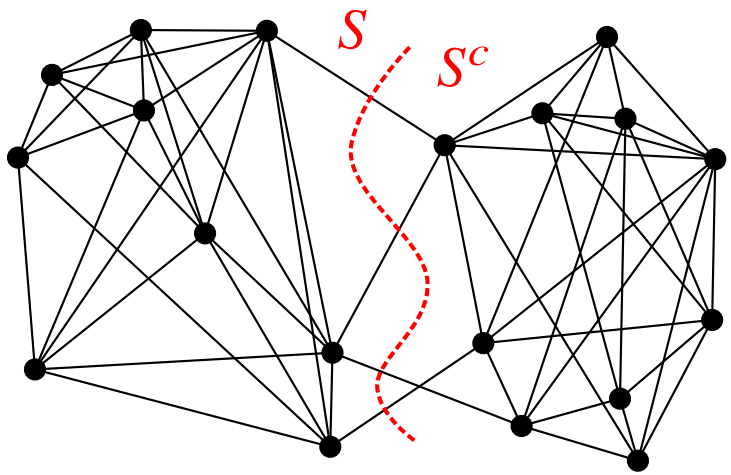
Can we get exact quantum algorithms?

Quantum algorithm for MIN CUT

back to MIN CUT:



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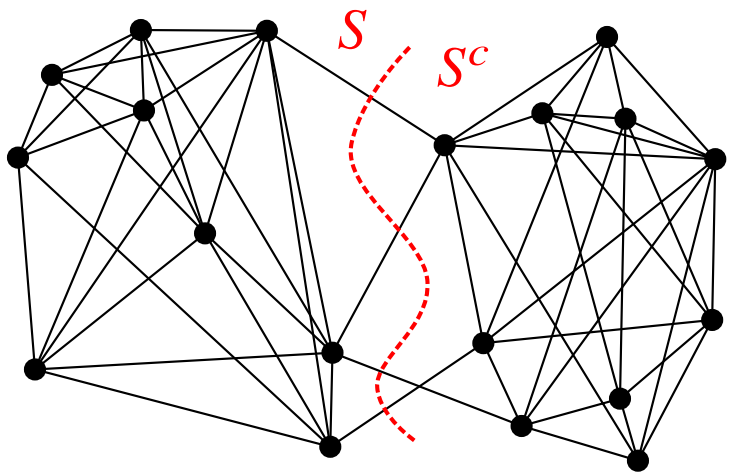

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quantum: ϵ-*approximate* MIN CUT in time $\widetilde{O}(\sqrt{mn}/\epsilon)$:(

Quantum algorithm for MIN CUT

input size = m input size = n^2

Results in adjacency *list* model and adjacency *matrix* model:

Theorem

Quantum algorithm for MIN CUT

Results in adjacency *list* model and adjacency *matrix* model:

Theorem

- *There exists a family of* ***weighted graphs*** *with* $\Omega(n^2)$ *edges for which the quantum query complexity of exact* MIN CUT *is* $\Omega(n^2)$ *in both models.* → no quantum speedup!

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- *For* ***unweighted graphs****, exact* MIN CUT *has quantum query complexity*
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tight

gap

Lower bounds for unweighted graphs (from connectivity [DHHM'06]):

$\Omega(n)$ (list) or $\Omega(n^{3/2})$ (matrix)

Quantum algorithm for MIN CUT

Upper bounds on quantum query complexity:

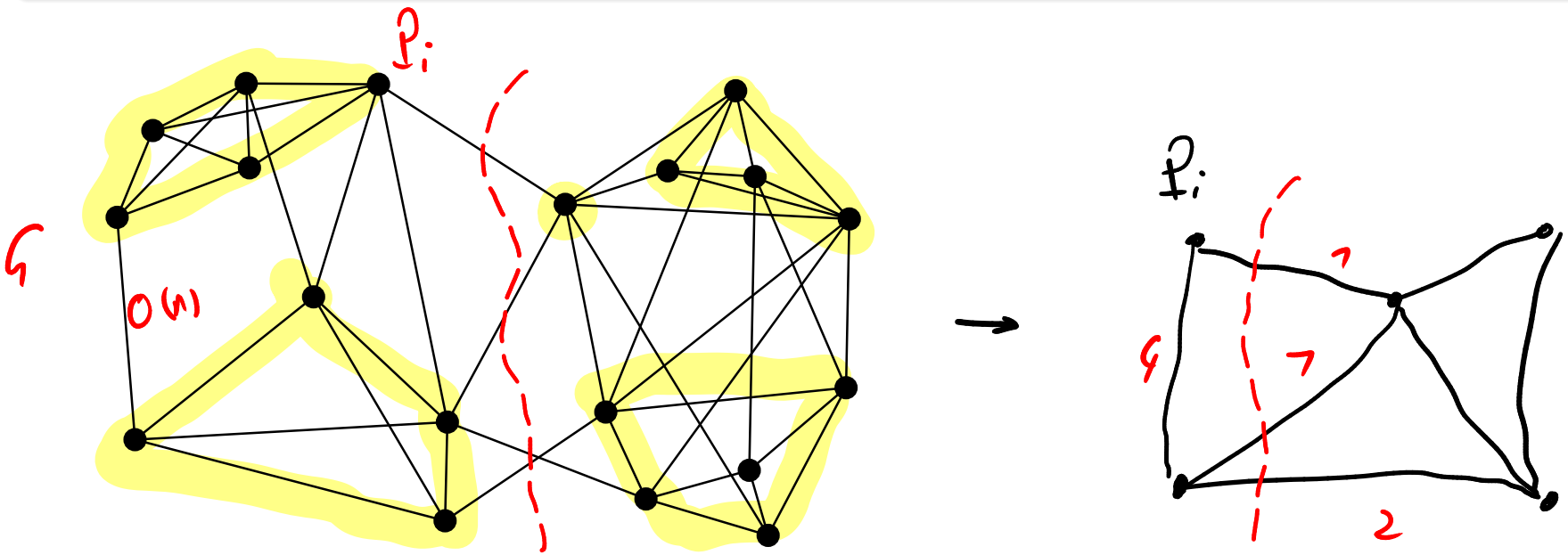
Quantum algorithm for MIN CUT

Upper bounds on quantum query complexity:

Theorem (Kawarabayashi-Thorup 2014, Rubinstein-Schramm-Weinberg 2017)

For unweighted G, there exists partition $V = P_1 \cup \cdots \cup P_k$ such that

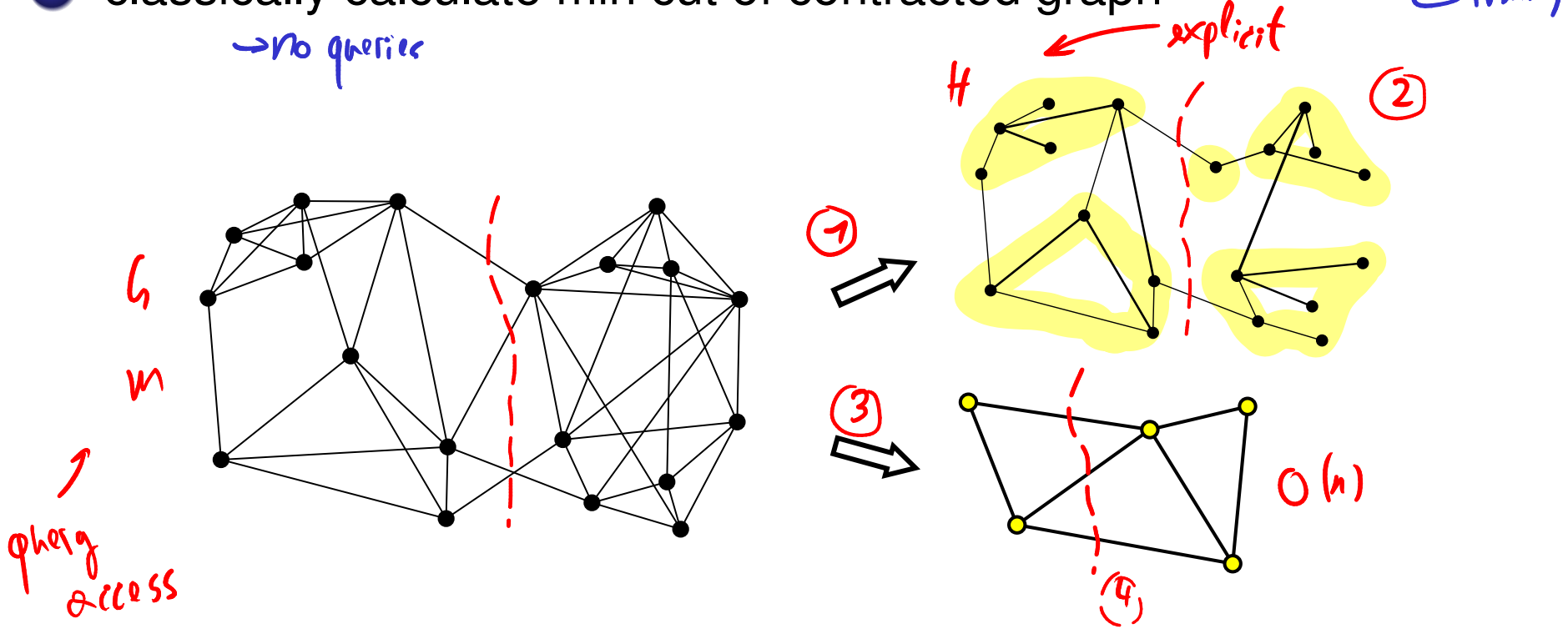
1. *there are $O(n)$ edges between partitions,*
2. *partition "respects" all near-minimum cuts.*



Quantum algorithm for MIN CUT

quantum query algorithm in list model:

1. construct $\epsilon = 1/10$-sparsifier H → $\tilde{O}(\sqrt{mn})$
   → min cut of G is near-minimum cut of H
2. learn partition $V = P_1 \cup \cdots \cup P_k$ of H → no queries
   → respects min cut of G
3. Grover search over m edges to find $O(n)$ edges *of* G between P_i's
4. classically calculate min cut of contracted graph → $O(\sqrt{mn})$



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  - $\widetilde{O}(n/\epsilon^2)$ for sparsification?
  - $\widetilde{O}(n)$ for unweighted min cut in adjacency list model?

$\Omega(n) \rightarrow \tilde{O}(\sqrt{mn})$

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thank you! stay safe!

$\widetilde{O}(n^{3/2}/\epsilon) < \widetilde{O}(n^2)$