Multi-port teleportation schemes.

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Port-based teleportation and its importance The port-based teleportation (PBT) protocol introduced in 2008 by Ishizaka and Hiroshima is a variant of quantum teleportation scheme which transmits the unknown state to the receiver without requiring any corrections on his/her side [1, 2] in the opposite to the standard setting presented in [3]. In the primary setting in this protocol, the sender and the receiver share N copies of the maximally entangled states (resource state), where each singlet is a two-qudit state, called port (see the left panel of Figure 1). The sender implements a joint POVM on the teleported system θ_C and the half of the resource state, getting an outcome $1 \leq i \leq N$ transmitted through a classical channel to the receiver. To recover the state the receiver just has to pick up the right port, pointed by classical message i. Depending on the type of POVM, one distinguishes two operational regimes: *probabilistic* and *deterministic*. In the former case, the measurement is designed to ensure that the teleported state arrives intact to the receiver, but there is a small probability of failure. In the latter case, the state always gets to the receiver but incurs some distortion, and its performance is described by fidelity computed between the input and the output state. We can also introduce the optimised versions of PBT. In this case Alice before measurements applies a global operation O_A on her half of the resource state resulting in sharing non-maximally entangled state. This boosts the performance of both variants of PBT, giving us square improvement in N. We will refer to those scenarios as *optimal PBT* (OPBT) and *non-optimal PBT*.

PBT protocols allow for entirely new applications in modern quantum information science. For instance, PBT has found its place in non-local quantum computations and position-based cryptography [4] resulted in new attacks on the cryptographic primitives, reducing the amount of consumable entanglement from doubly exponential to exponential, communication complexity [5] connecting the field of communication complexity and a Bell inequality violation, theory of universal programmable quantum processor performing computation by teleportation [1], universal simulator for qubit channels [6] improving simulations of the amplitude damping channel and allowing to obtain limitations of the fundamental nature for quantum channels discrimination [7]. Recently some aspects of PBT play a role in the general theory of identification of cause-effect relations [8], construction of universal quantum circuit for inverting general unitary operations [9] as well as theory of storage and retrieval of unitary quantum channels [10].

Problem: One could ask what happens if we wish to teleport using PBT a state of composite system or several systems, let us say k. One of the answer could be the following:

- Run the original PBT with dimension of the port equal to d^k, however the performance of the PBT protocols gets worse with growing local dimension [11, 12].
- We could also keep dimensions of the ports and split the resource state into k packages and then run k PBT procedures independently. However, here we need to reduce the number of shared entangled pairs between the parties, which possibly also reduces efficiency of the transfer.

Motivation: To send a large amount of quantum information more effectively than in the pre-existing methods described above. In the next part of this note, we argue that there is an efficient way to achieve the described goal.

OUR SOLUTION: Multiport based teleportation scheme

We propose a novel scheme - an extension of original PBT scheme, that allows to teleport many systems *in one go*. (see the right panel of Figure 1). We further obtain analytical formulas and useful bounds for performance of the proposed protocol.

- *Analytical formulas for deterministic scheme:*
  - Useful compact lower bound for entanglement fidelity F in non-optimal case [13] by considering equivalent task which is state discrimination problem [1, 4] and an exact expression for F, by exploiting tools

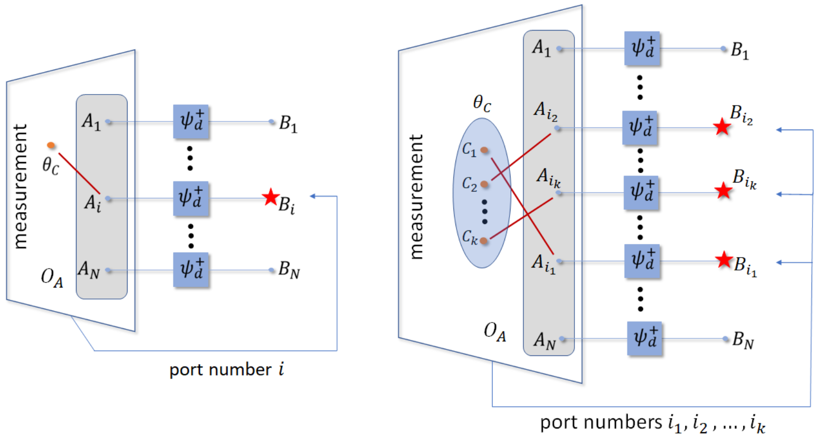


FIG. 1: On the left-hand side we present the scheme for the standard PBT, while on the right for multi-port based protocols. On the right-hand side we depict configuration for MPBT scheme. Two parties share N copies EPR pairs $\Phi_d^+ = |\psi_d^+\rangle\langle\psi_d^+|$, where $|\psi_d^+\rangle = (1/\sqrt{d})\sum_i |ii\rangle$. Alice to teleport an unknown joint state $\theta_C = \theta_{C_1C_2\cdots C_k}$, where $k \leq \lfloor N/2 \rfloor$, to Bob performs a global measurement (the blue trapeze) on systems $C_1 \cdots C_k A_1 \cdots A_N$, getting a classical outcome $\mathbf{i} = (i_1, i_2, \ldots, i_k)$. She transmits the outcome $\mathbf{i}$ via classical communication to Bob. The index $\mathbf{i}$ indicates on which k ports on the Bob's side the teleported state arrives (red stars). To recover the state he has permute selected ports according to index $\mathbf{i}$. In both versions by the grey rectangle we denote the optimal operation O_A applied by Alice to boost the performance of all variants of the protocols.

coming from representation theory [14]:

$$F \geq \binom{N}{k}\binom{d^2+N-1}{k}^{-1}, \qquad F = \frac{1}{d^{N+2k}} \sum_{\alpha \vdash N-k} \left(\sum_{\mu \in \alpha} m_{\mu/\alpha} \sqrt{m_\mu d_\mu} \right)^2, \tag{1}$$

where m_μ, d_μ denotes multiplicity and dimension of irreducible representations (irreps) of symmetric group $S(N)$ for which Young diagrams μ can be obtained from Young diagrams α of $N-k$ boxes by adding k boxes. By $m_{\mu/\alpha}$ we denote the number of possible ways when going from α to μ.

– Explicit and effective expression for F in the qubit case exploiting theory of angular momentum, see technical manuscript or [13]:

– Efficiently computable formula for fidelity in optimal scheme in terms maximal eigenvalue of a some matrix - generalisation of teleportation matrix M_F from [12] - which encodes the relationship between a set of Young diagrams and emerges as the optimal solution of the semidefinite program [15].

- *Analytical formulas in probabilistic scheme:*

  – Exact value of success probability in non-optimal case:

$$\text{qudits:} \quad p_{succ} = \frac{1}{d^N} \sum_{\alpha \vdash N-k} \min_{\mu \in \alpha} \frac{d_\mu}{m_\mu}, \qquad \text{qubits:} \quad p_{succ} = \frac{1}{2^N} \frac{1}{N+1} \sum_{s=0(\frac{1}{2})}^{\frac{N-k}{2}} (2s+1)^2 \binom{N+1}{\frac{N-k}{2} - s}. \tag{2}$$

  – Extremely simple formula for p_{succ} in optimal case [15]:

$$p_{succ} = \frac{N(N-1)\cdots(N-k+1)}{(d^2+N-1)(d^2+N-2)\cdots(d^2+N-k)} = \prod_{i=2}^{d^2} \left(1 - \frac{k}{N-1+i}\right). \tag{3}$$

APPLICATION. Asymptotic scaling of number of teleported particles: MPBT beats PBT.

We have applied the above formulas for performance of MPBT to obtain *qualitative* improvement of asymptotic "teleportation capacities" of MPBT scheme over the PBT based schemes. Specifically, we let the number of ports N grow to infinity, and consider teleportation of k qubits, with $k \simeq N^\alpha$, $0 \leq \alpha \leq 1$. In the limit

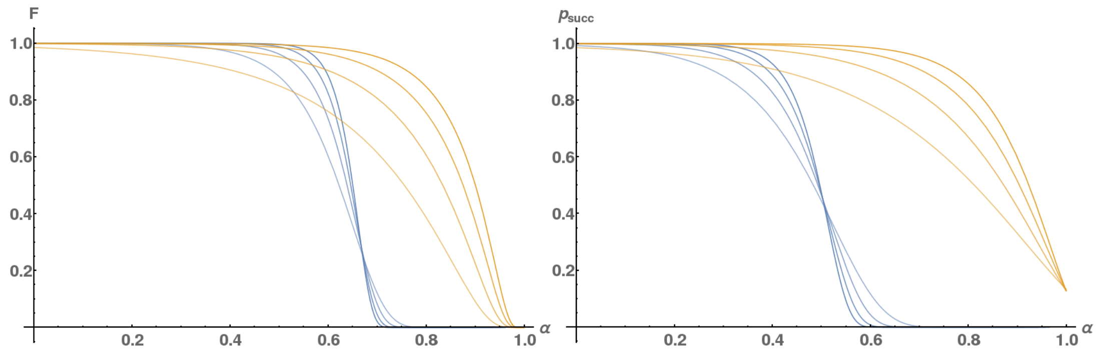
$N \to \infty$ we obtain sharp phase-like transitions in α, with faithful transmission below some threshold α_0, and completely flawed transmission above the threshold:

Deterministic scheme (see Figure 2a): even non-optimal MPBT beats optimal PBT.

– applying so-called packaged PBT, i.e. sending every particle via a package of N/k ports, we are able to teleport k particles with N ports using PBT. For $\alpha > 2/3$ fidelity is asymptotically zero.
– applying MPBT we obtain asymptotically faithful teleportation, i.e. $F \to 1$ for all $\alpha < 1$.

Probabilistic scheme (Figure 2b): optimal MPBT beats optimal PBT.

– Faithful teleportation (i.e. such that $p_{succ} \to 1$ for $N \to \infty$) is obtained both in OPBT and MPBT in the same region $\alpha < 1/2$; p_{succ} drops to zero for $\alpha < 1/2$.
– OMPBT given by (3) allows for significantly better teleportation: $p_{succ} \to 1$ for all $\alpha < 1$.



(a) (b)

FIG. 2: **Scaling of entanglement fidelity for transmission of $k = N^\alpha$ qubits.** (a) optimal PBT protocol (blue) compared with the bound for the fidelity in MPBT, given by (1) (yellow). (b) exact value of p_{succ} for OMPBT (3) (blue) compared with the one obtained by OPBT (yellow). Each four curves correspond to number of ports $N = 10^2, 10^3, 10^4$ and 10^5. We proved that the curves become step functions for $N \to \infty$.

MAIN TECHNICAL CONTRIBUTION

- We present *novel mathematical tools* concerning both standard Schur-Weyl duality [16] based on n-fold tensor product of unitary transformations, $U^{\otimes n}$, as well as, its "skew" version based on the product of type $U^{\otimes N} \otimes \overline{U}^{\otimes k}$ (bar denotes complex conjugation). Up to now the full characterisation of irreps for $\mathcal{A}_n^{(k)}(d)$ has been done for $k = 1$ and successfully applied to PBT [12, 17, 18]. However here, one has to solve far more complex problem - finding an orthonormal operator basis for $k > 1$. In particular, our contribution is:

  – We connect commutant of $U^{\otimes N} \otimes \overline{U}^{\otimes k}$ with algebra of the partially transposed permutation operators $\mathcal{A}_n^{(k)}(d)$ and identify ideal $\mathcal{M}^{(k)}$ crucial for solving MPBT.
  – We construct an irreducible orthonormal operator basis within $\mathcal{M}^{(k)}$ and investigate its properties.
  – We deliver expressions for evaluating partial trace over an arbitrary number of particles from group-theoretic objects (i.e. Young projectors), extending our previous knowledge in this direction.
  – Developing formalism of partially reduced irreducible representations (PRIR) - a new toolkit for efficient computations in operator basis of $\mathcal{M}^{(k)}$.

- These results are of the separate interests, since they play a role in:

  – New perspective in physics by studying: antiferromagnetic systems [19], gravity theories [20, 21] and particle physics [22] or quantum reference frame problems [23–26]
  – Modern applied mathematics: shows a connection between the above-mentioned physical applications and and Jucys-Murphy elements [27, 28] and a novel approach to representation theory of symmetric group presented in [29] (Fields medal in 2006), together with direction to its extensions.

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