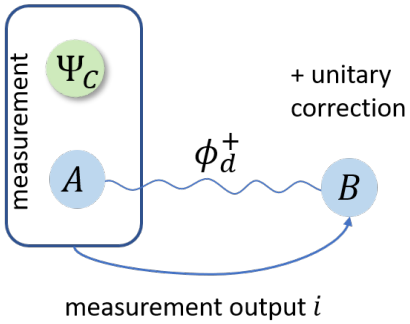
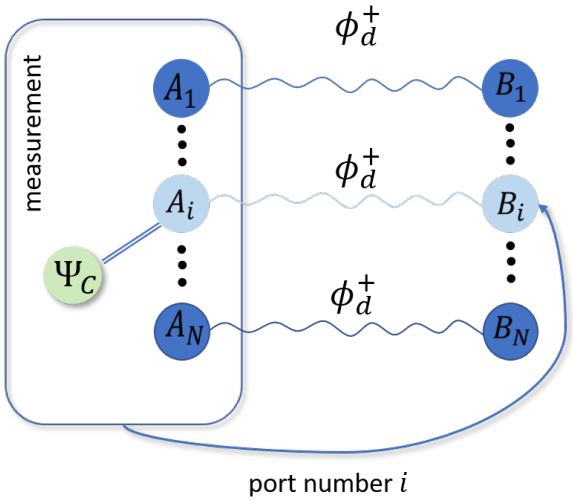
Multi-port teleportation schemes

Michał Studziński (University of Gdańsk)
Piotr Kopszak (University of Wrocław)

joint work with:
Michał Horodecki (ICTQT/University of Gdańsk),
Marek Mozrzymas (University of Wrocław)



PRL 70, 1895-1899 (1993)



For $d = 2$ sloved in:
PRL 101, 240501 (2008)
PRA 79, 042306 (2009)

For $d > 2$ solved in:
Sci. Rep. 7, 10877 (2017)
NJP 20.5, 053006 (2018)
CMP 381, 379-451 (2021)

Requires tools from representation theory of $S(n)$ and U.

Deterministic Scheme

- Set of measurements $\{\Pi_i\}_{i=1}^N$
- The state Ψ_C is always teleported, but imperfectly
- Performance is described by the *fidelity F*

Probabilistic Scheme

- Measurement Π_0 corresponding to failure
- The state Ψ_C is teleported perfectly
- Performance is described by the *probability of success* p_{succ}

Applications of PBT

- New architecture for the universal programmable quantum processor
  S. Ishizaka, T. Hiroshima, PRA **79**, *042306 (2009)*
- Efficient attacks for position based cryptography
  S.Beigi, R. König, NJP **13**, *093036 (2011)*
- Violation of Bell inequality from any large quantum advantage
  H. Buhrman et al. PNAS **113**, *3191-3196 (2016)*
- Fundamental limitations for quantum channels discrimination
  S. Pirandola et al. npj Quantum Information **5**, *50 (2019)*
- Universal simulator of quantum channels
  J. Pereira et al. arXiv:1912.10374

How many states can one faithfully teleport for a given numer of ports N?

PBT with large port dimension d^k

Sci. Rep. 7, 10877 (2017)
NJP 20.5, 053006 (2018)
CMP 381, 379-451 (2021)

Recycling protocol for PBT

PRL 110, 101505 (2013)
arXiv:2008.00856

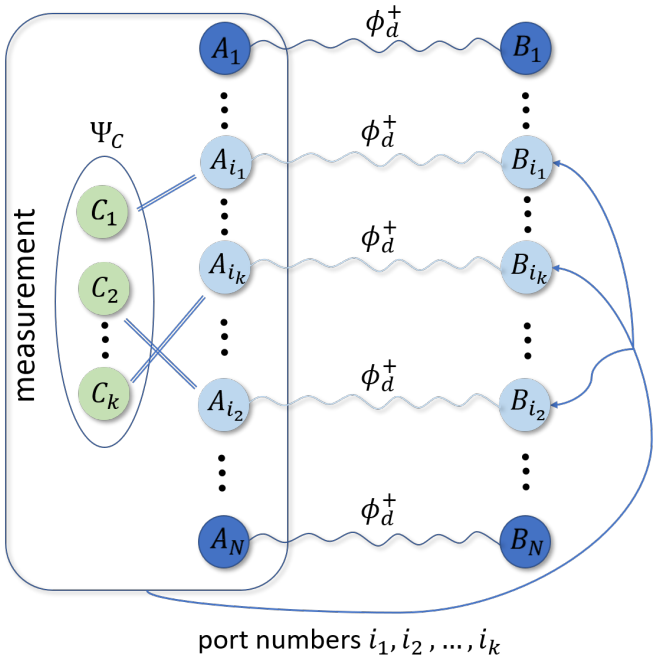
Packaged PBT

PRL 110, 101505 (2013)
arXiv:2008.00856

Multi-port-based protocol (MPBT)

PRL 110, 101505 (2013)
arXiv:2008.00856
arXiv:2008.00984
arXiv:2011.09256

Multi-port-based teleportation



PRL 110, 101505 (2013)
arXiv:2008.00856
arXiv:2008.00984
arXiv:2011.09256

- Structurally much more complicated than PBT
- Requires tools from representation theory of $S(n)$ and U

channel: $$\mathcal{N}(\Psi_C) = \sum_{\mathbf{i}\in\mathcal{I}} \mathrm{Tr}_{A\bar{B}_{\mathbf{i}}C}\left[\sqrt{\Pi_{\mathbf{i}}^{AC}}\left(\Phi_{AB}^{+}\otimes\Psi_C\right)\sqrt{\Pi_{\mathbf{i}}^{AC}}^{\dagger}\right]$$
$$= \sum_{\mathbf{i}\in\mathcal{I}} \mathrm{Tr}_{AC}\left[\Pi_{\mathbf{i}}^{AC}\left(\sigma_{\mathbf{i}}^{AB}\otimes\Psi_C\right)\right]$$

signals: $$\sigma_{\mathbf{i}}^{AB} \equiv \sigma_{\mathbf{i}} \equiv \mathrm{Tr}_{\bar{B}_{\mathbf{i}}}\Phi_{AB}^{+} = \frac{1}{d^{N-k}}\mathbf{1}_{\bar{A}_{\mathbf{i}}}\otimes\phi_{A_{i_1}B_{i_1}}^{+}\otimes\ldots\otimes\phi_{A_{i_k}B_{i_k}}^{+}$$

Multi-port-based teleportation

Problem 1: Evaluate efficiency of the protocol

1. Deterministic scheme (non-optimal + optimal case)

$$F = \mathrm{Tr}\left[\Phi^{+}_{BD}(\mathcal{N} \otimes \mathbf{1}_D)\left(\Phi^{+}_{CD}\right)\right] = \frac{1}{d^{2k}} \sum_{\mathbf{i}\in\mathcal{I}} \mathrm{Tr}(\Pi_{\mathbf{i}}\sigma_{\mathbf{i}}).$$

   In non-optimal we use square-root measurements (SRM):

$$\Pi_{\mathbf{i}} = \rho^{-1/2}\sigma_{\mathbf{i}}\rho^{-1/2} \quad \text{with} \quad \rho = \sum_{\mathbf{i}\in\mathcal{I}} \sigma_i.$$

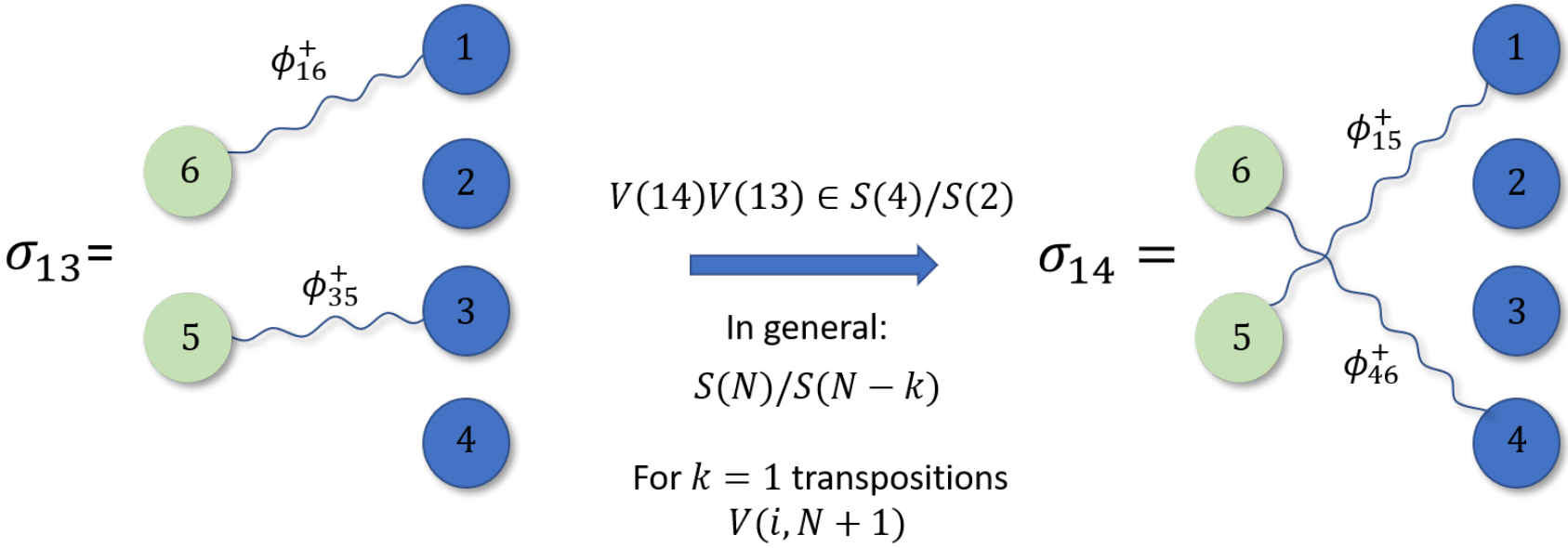
2. Probabilistic scheme (non-optimal + optimal case)

$$p_{succ} = \frac{1}{d^{N+k}} \sum_{\mathbf{i}\in\mathcal{I}} \mathrm{Tr}\left[\Pi_{\mathbf{i}}\right]$$

3. In optimal protocols Alice optimises simultaneously over shared MES and measurements.

We have two types of symmetries.

Consider $N = 4, k = 2$, so we have $2!\binom{4}{2} = 12$ signals $\sigma_{\mathbf{i}}$.



There is well known property:

$$|\psi^+_d\rangle = \frac{1}{\sqrt{d}} \sum_k |k\rangle \otimes |k\rangle \quad \text{is} \quad U \otimes U^* \text{ invariant}$$

In particular, we have:

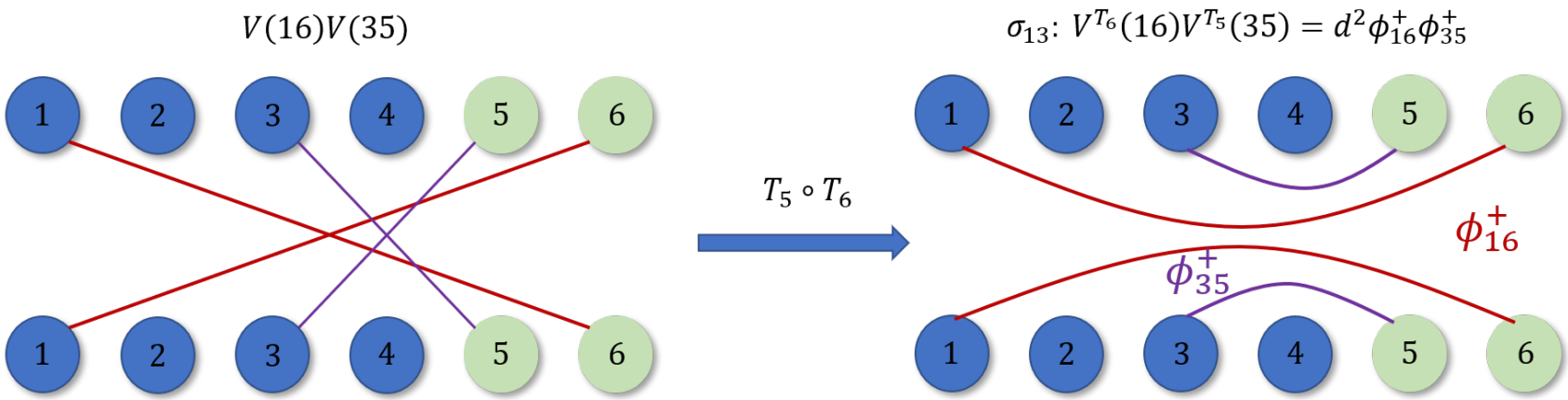
$$\sigma_{13}, \sigma_{14} \quad \text{are} \quad U^{\otimes 4} \otimes U^* \otimes U^* \text{ invariant}$$

1. Measurements $\Pi_{\mathbf{i}}$ and signals $\sigma_{\mathbf{i}}$ satisfy:

$$\text{invariance:} \quad U \otimes \cdots \otimes U \otimes \underbrace{U^* \otimes \cdots \otimes U^*}_{k}$$

$$\text{covariance:} \quad S(N)/S(N-k)$$

2. Complex conjugation translates to partial transposition: $V(1n)^{t_n} = dP_+ = d|\psi_d^+\rangle\langle\psi_d^+|_{1n} \otimes \mathbf{1}$.



Elements of Walled-Brauer Algebra
Annals of Mathematics 38 4, 857 (1937)
Journal of Algebra 320, 169 (2008)

Conclusion 1: $U \otimes \cdots \otimes U \otimes \underbrace{U^* \otimes \cdots \otimes U^*}_{k} \longleftrightarrow \mathcal{A}_d^{(k)}(n)$

Having $\mathcal{A}_n(d) := \mathrm{Span}_{\mathbb{C}}\{V(\sigma) : \sigma \in S(n)\} \subset \mathrm{Hom}((\mathbb{C}^d)^{\otimes n})$

$$V(\sigma)|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle = |i_{\sigma^{-1}(1)}\rangle \otimes |i_{\sigma^{-1}(2)}\rangle \otimes \cdots \otimes |i_{\sigma^{-1}(n)}\rangle$$

we define

$$\mathcal{A}_d^{(k)}(n) := \mathrm{Span}_{\mathbb{C}}\{V^{(k)}(\sigma) : \sigma \in S(n)\} \subset \mathrm{Hom}((\mathbb{C}^d)^{\otimes n})$$

Conclusion 2: Measurements $\Pi_{\mathbf{i}}$ and signals $\sigma_{\mathbf{i}}$ are elements of $\mathcal{A}_d^{(k)}(n)$ and are covariant w.r.t. to coset $S(N)/S(N-k)$.

Problem 2: **Find irreducible matrix representations of $\mathcal{A}_d^{(k)}(n)$.**

Solution in: arXiv:2008.00984

Our results: technical aspects

Structurally different algebras:

$\mathcal{A}_n(d) : V(an)V(an) = \mathbf{1}$; $\mathcal{A}_d^{(k)}(n) : V(an)^{t_n}V(an)^{t_n} = dV(an)^{t_n}$

We are motivated by **Schur-Weyl duality** for U and $S(n)$:

- actions of $U^{\otimes n}$ and $V(\pi)$ commutes ($\pi \in S(n)$)
- there exists a basis in which

dimension of irrep λ

multiplicity of irrep λ

$$\mathcal{H} = \underbrace{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d}_{n} \simeq \bigoplus_\lambda \underbrace{\mathbb{C}^{d_\lambda}}_{\text{representation space of irrep } \lambda} \otimes \underbrace{\mathbb{C}^{m_\lambda}}_{\text{multiplicity space of irrep } \lambda}$$

λ -labels irreps of $S(n)$; Young diagrams of n boxes

Our results: technical aspects

- Within every irrep λ of $S(n)$

$$E^{\lambda}_{ij} = \frac{d_\lambda}{n!} \sum_{\pi \in S(n)} \varphi^{\lambda}_{ji}(\pi^{-1}) V(\pi), \qquad P_\lambda = \sum_{i=1}^{d_\lambda} E^{\lambda}_{ii}$$

- We have constructed similar operators for commuting actions of $U^{\otimes N} \otimes (U^*)^{\otimes k}$ and $V^{(k)}(\pi)$.
- Our irreps are labelled by irreps α of $S(N-k)$ and irreps μ of $S(N) \longrightarrow F_\mu(\alpha)$, for $\mu \in \alpha$.

$$\Pi_{\mathbf{i}} = \frac{1}{\sqrt{\rho}} \sigma_{\mathbf{i}} \frac{1}{\sqrt{\rho}}, \quad \rho = \sum_{\alpha} \sum_{\mu \in \alpha} \lambda_\mu(\alpha) F_\mu(\alpha), \quad \lambda_\mu(\alpha) = \frac{k! \binom{N}{k}}{d^N} \frac{m_\mu}{m_\alpha} \frac{d_\alpha}{d_\mu}$$

$$\rho^{-1} = \sum_{\alpha} \sum_{\mu \in \alpha} \lambda_\mu^{-1}(\alpha) F_\mu(\alpha)$$

- Representation theory approach + semidefinite programming (SDP)

Our results: Deterministic scheme

- Maximally entangled states + SRM

$$F = \frac{1}{d^{N+2k}} \sum_{\alpha \vdash N-k} \left(\sum_{\mu \in \alpha} m_{\mu/\alpha} \sqrt{m_\mu d_\mu} \right)^2$$

- In particular, for $d = 2$, using spin-j angular momentum representation

$$F = \frac{1}{2^{N+2k}} \sum_{s=0(\frac{1}{2})}^{\frac{N-k}{2}} \left(\sum_{j=\max\{0(\frac{1}{2}), s-\frac{k}{2}\}}^{s+\frac{k}{2}} \frac{2j+1}{N+1} \sqrt{\binom{N+1}{\frac{N}{2}-j}} h_{sjk} \right)^2,$$

where

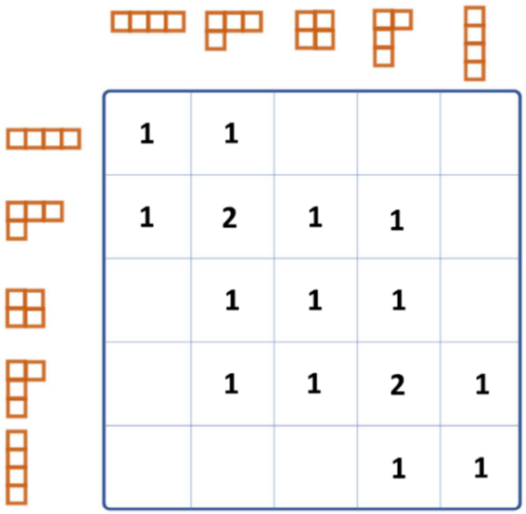
$$h_{sjk} = \binom{k}{s-j+k/2} - \binom{k}{s+j+k/2+1}.$$

Our results: Generalised teleportation matrix

Optimisation over Alice's measurements and entangled states

$$F = \frac{1}{d^{2k}} ||M_F^{d,k}||_\infty \equiv \frac{1}{d^{2k}} \lambda_{\max}(M_F^{d,k}),$$

where $M_F^{d,k}$ is generalised teleportation matrix.



1	1			
1	2	1	1	
	1	1	1	
	1	1	2	1
			1	1

$N = 4, k = 1, d \geq 4$

1	3	3	1	0	0	0	0	0	0	0
3	10	12	6	1	0	0	0	0	0	0
3	12	19	15	6	1	0	0	0	0	0
1	6	15	20	15	6	1	0	0	0	0
0	1	6	15	20	15	6	1	0	0	0
0	0	1	6	15	20	15	6	1	0	0
0	0	0	1	6	15	20	15	6	1	0
0	0	0	0	1	6	15	20	15	6	1
0	0	0	0	0	1	6	15	20	15	5
0	0	0	0	0	0	1	6	15	19	9
0	0	0	0	0	0	0	1	5	9	5

$N = 20, k = 3, d = 2$

Our results: Probabilistic scheme

- Optimisation over entangled states:

$$p_{succ} = \frac{1}{d^N} \sum_{\alpha \vdash N-k} m_\alpha^2 \min_{\mu \in \alpha} \frac{d_\mu}{m_\mu}$$

In particular, for $d = 2$, using spin-j angular momentum representation

$$p_{succ} = \frac{1}{2^N} \sum_{s=0(\frac{1}{2})}^{\frac{N-k}{2}} \frac{1}{N+1} \binom{N+1}{\frac{N-k}{2} - s}$$

- Optimisation over Alice's measurements and entangled states

$$p_{succ} = \frac{N!}{(N-k)!} \frac{(d^2+N-k-1)!}{(d^2+N-1)!}$$

PART II

Bound on F from state discrimination

- The fidelity relates to probability of discrimination

$$F = \frac{k!\binom{N}{k}}{d^{2k}} p_{disc} = \frac{1}{d^{2k}} \sum_{\mathbf{i}\in\mathcal{I}} \mathrm{Tr}(\Pi_{\mathbf{i}}\sigma_{\mathbf{i}}) \tag{1}$$

of the ensemble $\mathcal{E} \equiv \{1/(k!\binom{N}{k}), \sigma_{\mathbf{i}}\}$.

- From the bound (König, Beigi, NJP 13, 093036 (2011))

$$p_{disc} \geq \frac{1}{d^{N-k} k!\binom{N}{k} \mathrm{Tr}\,\overline{\rho}^2}$$

we have

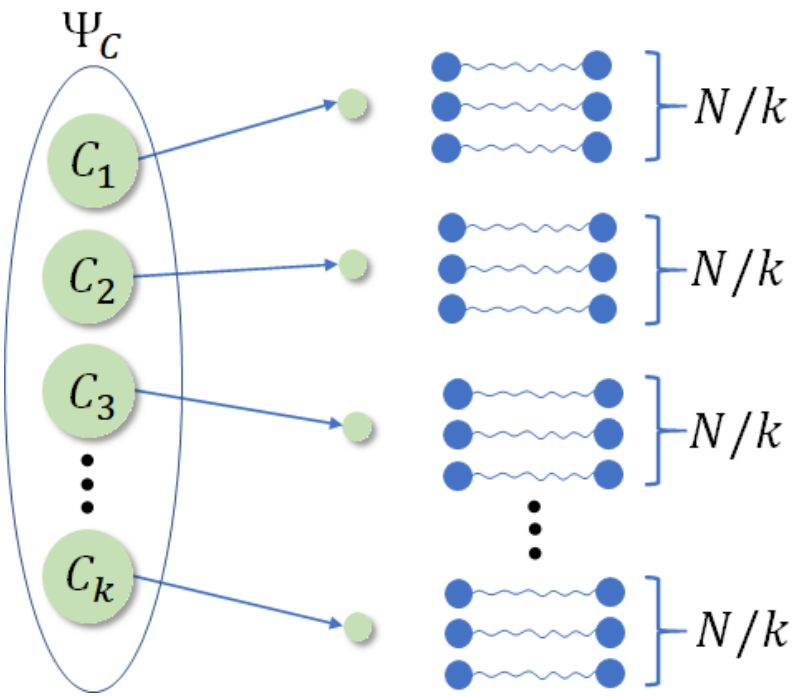
$$F \geq \binom{N}{k}\binom{d^2+N-1}{k}^{-1} \geq \left(1 - \frac{d^2-1}{d^2+N-2k}\right)^k .$$

Comparing the bound for fixed k

Comparing lower bound for F in MPBT protocol against:

- standard PBT with $k = 1$, $d_{PBT} = d^k \longrightarrow$ poor performance due to large dimension
- lower bound for recycling protocol $\longrightarrow$ similar behaviour, but the bound is weaker
- packaged *PBT*, where ports are divided into k packages employing standard *PBT* $\longrightarrow$ interesting behaviour

Packaged OPBT (Strelchuk, M. Horodecki and Oppenheim, PRL 110, 010505 (2013))



- Ports divided into k packages
- Define

$$F_{pack}(N,k) := F(N/k,1)^k$$

Comparision between the lower bound and the packaged OPBT

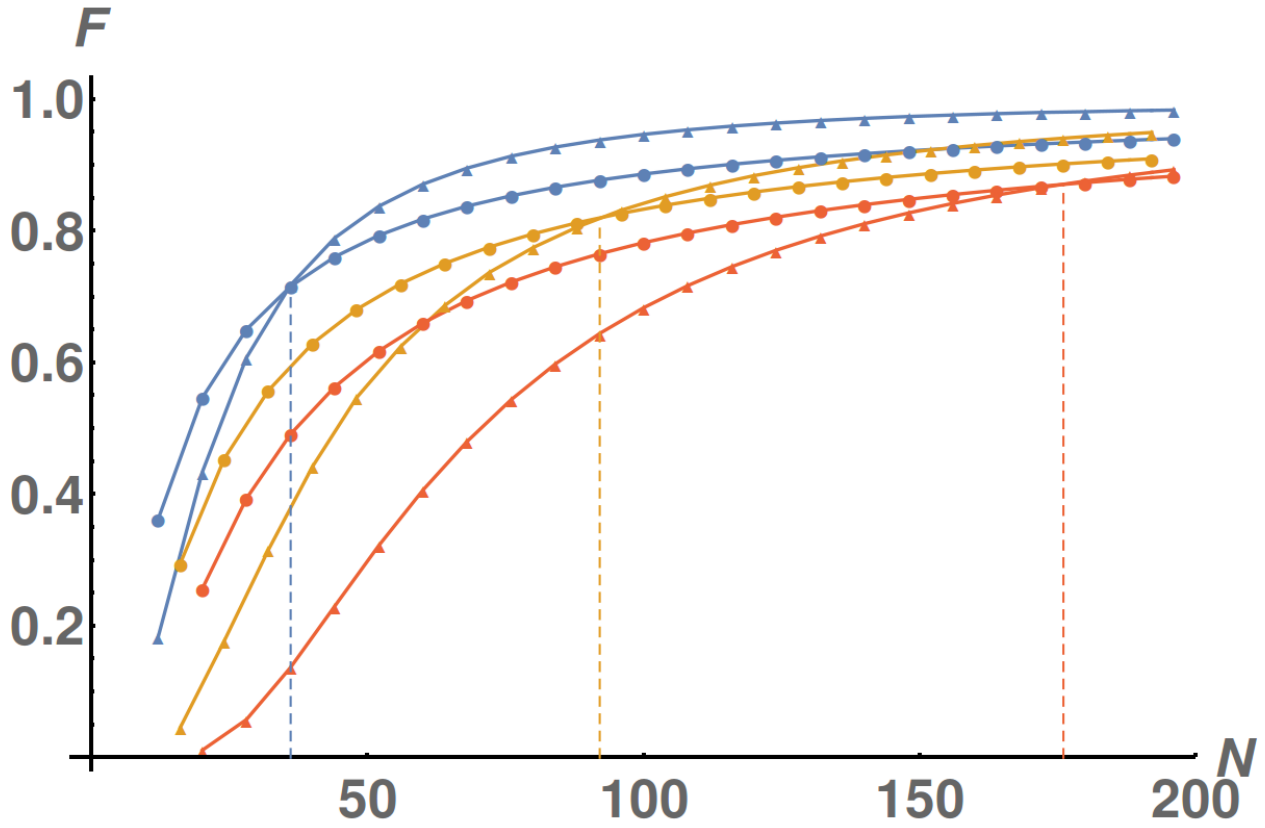


Figure: Effective lower bound for fidelity in multi-port based teleportation scheme (full circles) vs packaged OPBT (triangles) for fixed k=4, 6, 8 (blue, orange and red) for $d = 2$.

Changing number of teleported qudits adaptively

Let

$$k \equiv k(N) = aN^{\alpha},\ \alpha \in (0, 1]$$

The exact value for F in OPBT protocol is (Ishizaka, Hiroshima, PRA 79, 042306 (2009))

$$F = \cos^2\left(\frac{\pi}{N+2}\right) \approx 1 - \frac{\pi^2}{N^2}$$

thus

$$F_{pack}^{OPBT}(N, k) = \left(1 - \frac{a^3\pi^2}{aN^{2(1-\alpha)}}\right)^{aN^{\alpha}}$$

and

$$\lim_{N\to\infty} F_{pack}^{OPBT}(N, k) = \begin{cases} 0, & \alpha > \frac{2}{3}, \\ e^{-\pi^2 a^3}, & \alpha = \frac{2}{3}, \\ 1, & \alpha < \frac{2}{3}. \end{cases}$$

Asymptotic behaviour of lower bound on F in *MPBT*

- On the other hand, applying Bernoulli inequality

$$(1-x)^n > 1 - nx, \ x < 1$$

to the weaker bound in MPBT protocol, we get

$$\left(1 - \frac{d^2-1}{d^2+N-2k(N)}\right)^{k(N)} \geq 1 - k(N)\frac{d^2-1}{d^2+N-2k(N)}$$

which goes to 1 if $k = o(N)$.

- When $k = aN, \ a < \frac{1}{2}$

$$\lim_{N\to\infty}\left(1 - \frac{d^2-1}{d^2+N-2k(N)}\right)^{k(N)} = e^{-\frac{(d^2-1)a}{1-2a}}$$

Comparision of asymptotic behaviur of *MPBT* and *packaged OPBT*

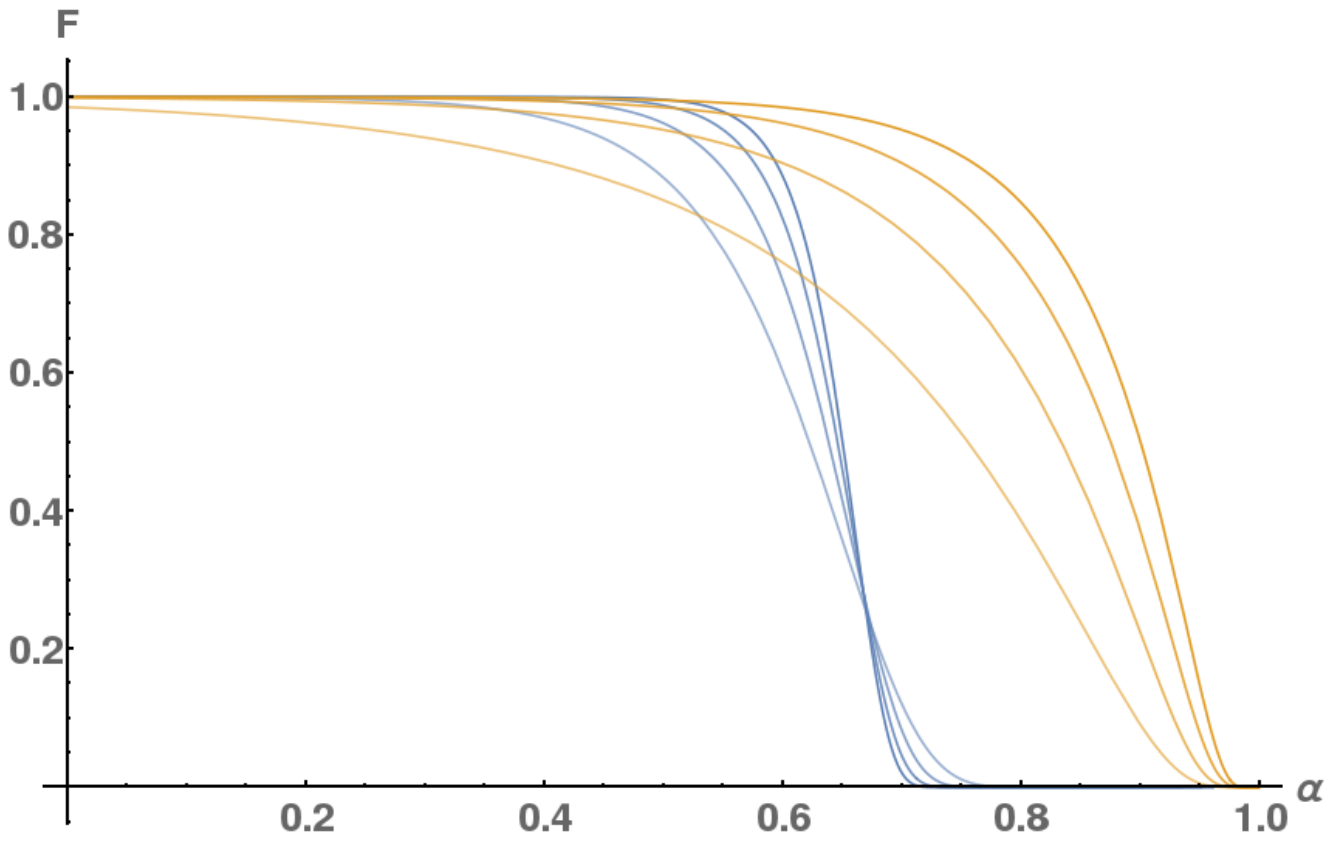


Figure: Entanglement fidelity compared for OPBT (blue lines) and MPBT (orange lines) protocols, where $k = aN^{\alpha}$, $a = \frac{1}{2}$. N runs through 10^2, 10^3, 10^4 and 10^5 as the lines become thicker.

Recycling protocol (Strelchuk, Horodecki and Oppenheim, PRL 110, 010505 (2013))

Similar comparison can be made with the *recycling protocol*, where resource states can be used multiple times

- Alice teleports 1st state to port z
- Bob performs SWAP of zth and $1st$ ports
- $1st$ port is marked
- A and B repeat above steps with unmarked ports

The lower bound for the fidelity in the recycling protocol for qubits:

$$F(N,k) \geq 1 - \frac{11k}{2N}$$

Comparision of different protocols in the asymptotic regime

	$F=0$	F_{cr}	$F=1$
PBT	$\alpha > 1/2$	$\alpha_{\mathrm{cr}} = 1/2\,,\, F_{\mathrm{cr}} = e^{-3a^2/4}$	$\alpha < 1/2$
OPBT	$\alpha > 2/3$	$\alpha_{\mathrm{cr}} = 2/3\,,\, F_{\mathrm{cr}} = e^{-\pi a^3}$	$\alpha < 2/3$
MPBT	$-$	$\alpha_{\mathrm{cr}} = 1, \quad F_{\mathrm{cr}} \geq e^{-\frac{(d^2-1)a}{1-2a}}$	$\alpha < 1$
recycling	$-$	$\alpha_{\mathrm{cr}} = 1, \quad F_{\mathrm{cr}} \geq 1 - \frac{11a}{2}$	$\alpha < 1$

Table: The comparison of different regions corresponding to different asymptotic behaviours. Although *MPBT* and *recycling* regions coincide, the former is usable whenever $a < 1/2$ as opposed to $a < 1/11$ in the latter case

Central Limit Theorem for p_{succ}

- Basing on

$$p_{succ}(N,k) = \frac{1}{2^N} \sum_{s=0(\frac{1}{2})}^{\frac{N-k}{2}} \frac{1}{N+1} \binom{N+1}{\frac{N-k}{2} - s}.$$

we can express limiting value of p_{succ} as

$$\lim_{N\to\infty} p_{succ} = \begin{cases} 2\int_0^\infty x^2 \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(x+a)^2}{2}}\, dx, & k = a\sqrt{N}, a \in (0,1) \\ 2\int_0^\infty x^2 \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}\, dx = 1, & k = o(\sqrt{N}) \end{cases} \tag{2}$$

- The critical α is $1/2$ -same as in packaged OPBT.
- Only after introducing Optimal MPBT, with it's remarkably simple expression for p_{succ} we can outperform origtinal schemes.

p_{succ} in OMPBT

The expression, although general, is remarkably simple

$$p_{succ}^{OMPBT} = \prod_{m=2}^{d^2} \left(1 - \frac{k}{N-1+m}\right).$$

For $d = 2$

$$\lim_{N\to\infty} p_{succ}^{OMPBT} = \begin{cases} (1-a)^3, & k = aN \\ 1, & k = o(N) \end{cases} \tag{3}$$

which outperforms the packaged OPBT.

Final comparison for probabilistic protocol

	$p_s = 0$	$p_{s,\mathrm{cr}}$	$p_s = 1$
PBT	$\alpha > 1/3$	$\alpha_{\mathrm{cr}} = 1/3\,,\ p_{s,\mathrm{cr}} = e^{-ca^{3/2}}$	$\alpha < 1/3$
OPBT	$\alpha > 1/2$	$\alpha_{\mathrm{cr}} = 1/2,\ p_{s,\mathrm{cr}} = e^{-3a^2}$	$\alpha < 1/2$
MPBT	$\alpha > 1/2$	$\alpha_{\mathrm{cr}} = 1/2, p_{s,\mathrm{cr}} = \int_0^\infty \frac{2x^2}{\sqrt{2\pi}}\, e^{-\frac{(x+a)^2}{2}}$	$\alpha > 1/2$
OMPBT	$-$	$\alpha_{\mathrm{cr}} = 1,\quad p_{s,\mathrm{cr}} = (1-a)^3$	$\alpha < 1$

Table: Table collects a comparison of the asymptotic behaviour of all variants of packaged PBT with MPBT in probabilistic version. Here $c = \sqrt{8/\pi}$.

Open problems

There are still open problems regarding PBT and MPBT protocols:

- Find asymptotic expression for F in case of OMPBT
- Find more friendly expression for F in case of PBT and MPBT
- Try to simplify measurements in all variants of PBT keeping good performance
- More general analysis of the recycling protocol

References:

Port-based teleportation protocols for qubits:

- *S. Ishizaka, T. Hiroshima, PRL* **101**, *240501 (2008)*
- *S. Ishizaka, T. Hiroshima, PRA* **79**, *042306 (2009)*

Port-based teleportation protocols in arbitrary dimension:

- *M. Studziński et al. Sci. Rep.* **7**, *10871 (2017)*
- *M. Mozrzymas et al. NJP* **20.5**, *053006 (2018)*
- *M. Christandl et al. CMP* **381**, *379–451 (2021)*
- *F. Leditzky arXiv:quant-ph/2008.11194*

Multi-port based teleportation in arbitrary dimension:

- *S. Strelchuk et al. PRL* **110**, *069902 (2013)*
- *P. Kopszak et al. arXiv:quant-ph/2008.00856*
- *M. Studziński et al. arXiv:quant-ph/2008.00984*
- *M. Mozrzymas et al. arXiv:quant-ph/2011.09256*