Implementation of quantum measurements using classical resources and only a single ancillary qubit

Tanmay Singal, **Filip Maciejewski**
& Michał Oszmaniec

Center for Theoretical Physics
Polish Academy of Sciences

European Funds Smart Growth
Republic of Poland
Foundation for Polish Science
European Union European Regional Development Fund

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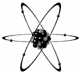
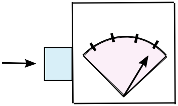
- Analysis of noisy implementation.
- Conclusion.

Measurements in quantum mechanics



ρ



$\{M_i\}_{i=1}^{N}$

Born rule

$$p(i|\rho) = \mathrm{tr}(M_i \rho)$$

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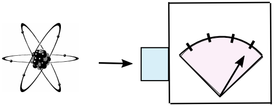

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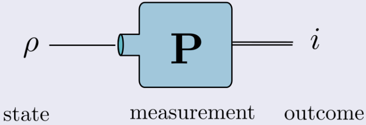
Applications

- Quantum communication, e.g., quantum state discrimination[1]
- Quantum metrology[2]
- Quantum tomography [3]
- Quantum computation, e.g. hidden subgroup problem [4]

[1] JMO 57(3) 160–180 (2010)
[2] J. Phys. A: Math. Gen. 47(42) 424006 (2014)
[3] RMP 89 035002 (2017)
[4] CJCTS 06 Vol 2006 (2006)

Positive **O**perator **V**alued **Measure** (POVM)

Projective measurements (simpler)



Description: $P_i \geq 0$, $\sum_i P_i = \mathbb{1}$, $P_i P_j = \delta_{ij}$

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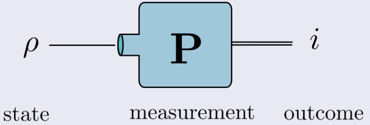

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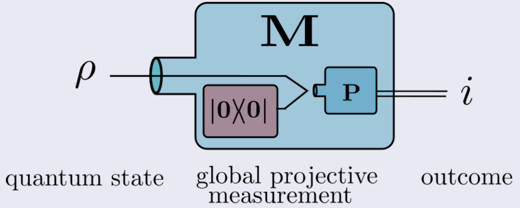
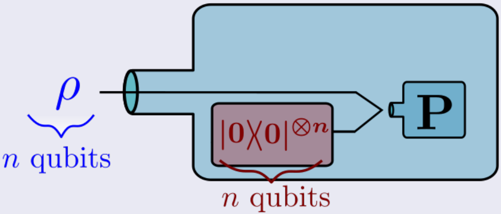
General measurements aka POVMs (Naimark's dilation theorem)



Description: $\mathbf{M} = (M_1, M_2, \cdots, M_n)$, $M_i \geq 0$, $\sum_i M_i = \mathbb{1}$

Resources needed to implement general POVMs

Generalized Naimark dilation theorem (PRL 119, 190501 (2017))



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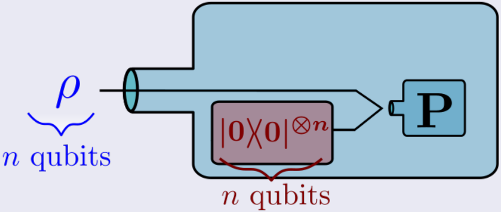

Experimental challenges (especially on NISQ!)

- More ancillary qubits $\Rightarrow$ larger circuit width $\Rightarrow$ more noise
- Limitations to qubit connectivity in NISQ.

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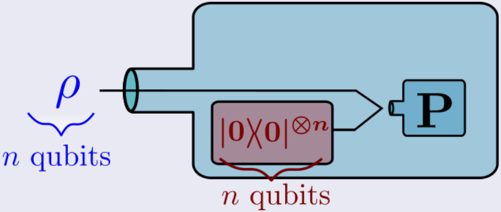

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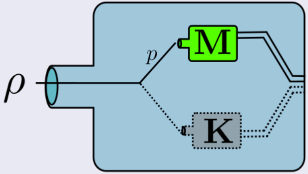
Motivation

Find ways to implement general POVM with fewer quantum resources (ancillary qubits)

Classical resources at our disposal

If **classical randomness** is free, then following operations are free[1].

Free Operations



Classical mixing
probabilistic mixture of POVMs

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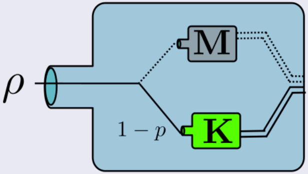

$$\mathbf{L} = p\mathbf{M} + (1-p)\mathbf{K}$$

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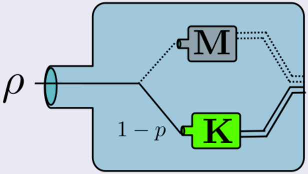

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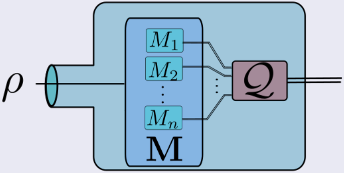

$$\mathcal{Q}(\mathbf{M})$$

Classical postprocessing

coarse – graining over outcomes

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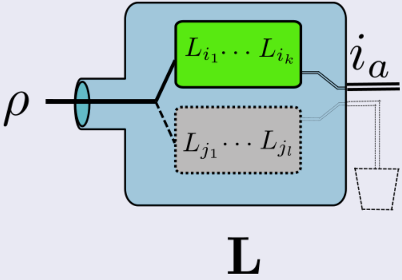
Post-Selection: another POVM operation

General post-selection operation



Outcomes of $\mathbf{L}$ partitioned into two sets: $i_1, \cdots, i_k$, and $j_1, \cdots, j_l$.
Sample from i_a's, neglect j_a's.

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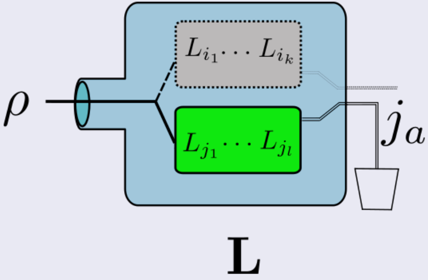

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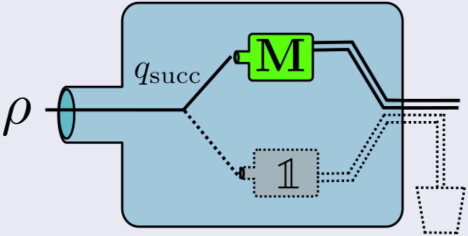
Simulating $\mathbf{M}$ by post-selection on $\mathbf{L}$



$$\mathbf{L} = \mathrm{q}_{\text{succ}}(\mathbf{M}, 0) + (1 - \mathrm{q}_{\text{succ}})(\mathbf{0}_n, \mathbb{1})$$

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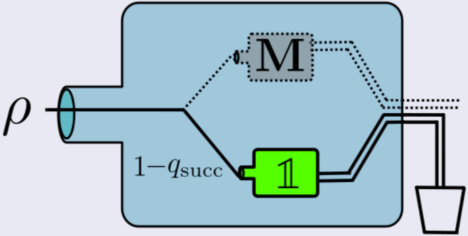

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Target POVM: $\mathbf{M} = (M_1, M_2, \cdots, M_n)$ $\mathrm{rank} M_i = 1$.

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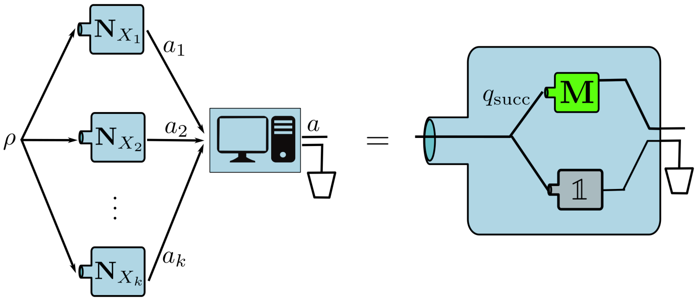
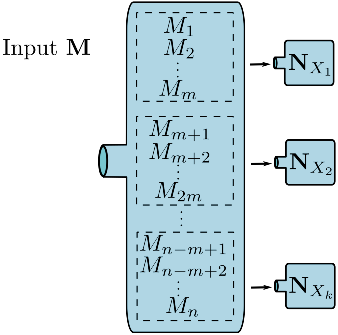


The scheme: randomly implement $\mathbf{N}_{X_j}$'s with fixed probability. Then coarse-grain over certain outcomes.
Using post-selection, $\mathbf{M}$ is implemented with success probability $\mathrm{q}_{\rm succ}$.

Scheme using classical resources and one ancillary qubit II

Target POVM: $\mathbf{M} = (M_1, M_2, \cdots, M_n)$ $\quad \text{rank} M_i = 1$

Actually, $\mathbf{N}_{X_i}$'s are constructed using M_i's.



Constraint on m so that $\mathbf{N}_{X_j}$ are implementable using a single ancillary qubit: $m \leq d$.

Scheme using classical resources and one ancillary qubit II

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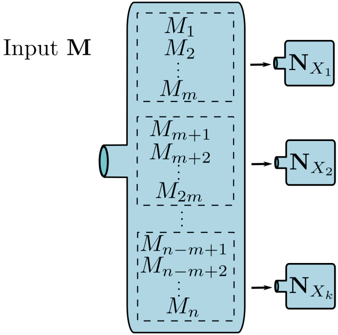

$$\text{Success probability}: \; q_{\text{succ}} = \left(\sum_{j=1}^{n/m} \left\| \sum_{i \in X_j} M_i \right\| \right)^{-1} .$$

A little bit more about $\mathrm{q_{succ}}$

$$q_{\mathrm{succ}}\left[\{X_j\}\right] = \left(\sum_{j=1}^{n/m} \left\| \sum_{i \in X_j} M_i \right\| \right)^{-1} .$$

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- $\mathrm{q}_{\mathrm{succ}}$ is related to other resource-theoretic quantities of $\mathbf{M}$ (will see in a while).

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Earlier work along the same lines: PRA 100, 012351 (2019)

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Now over to Filip for the numerics.

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Numerical interlude

Tanmay Singal, Filip Maciejewski, Michał Oszmaniec

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24th Annual Conference on Quantum Information Processing (QIP 2021)

Success probability of implementation

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$$q_{\mathrm{succ}}\left(\{X_j\}\right)$$

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- Problem (?): finding good partition could in principle be very hard…
- How it looks in practice?

$$q_{\text{succ}}(\{X_j\}) = \left(\sum_{X_j} \|\sum_{i\in X_j} M_i\|\right)^{-1}$$

Tested measurements

- We tested three classes of rank-1 POVMs with $n = d^2$ outcomes:

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Informationally complete

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$|\psi^0\rangle$

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$$M_g \approx |\psi^g\rangle \langle \psi^g|$$

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3. Symmetric Informationally Complete (SIC).

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success probability

X axis:
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dimensions range :

- random: d $\leq$ 1024
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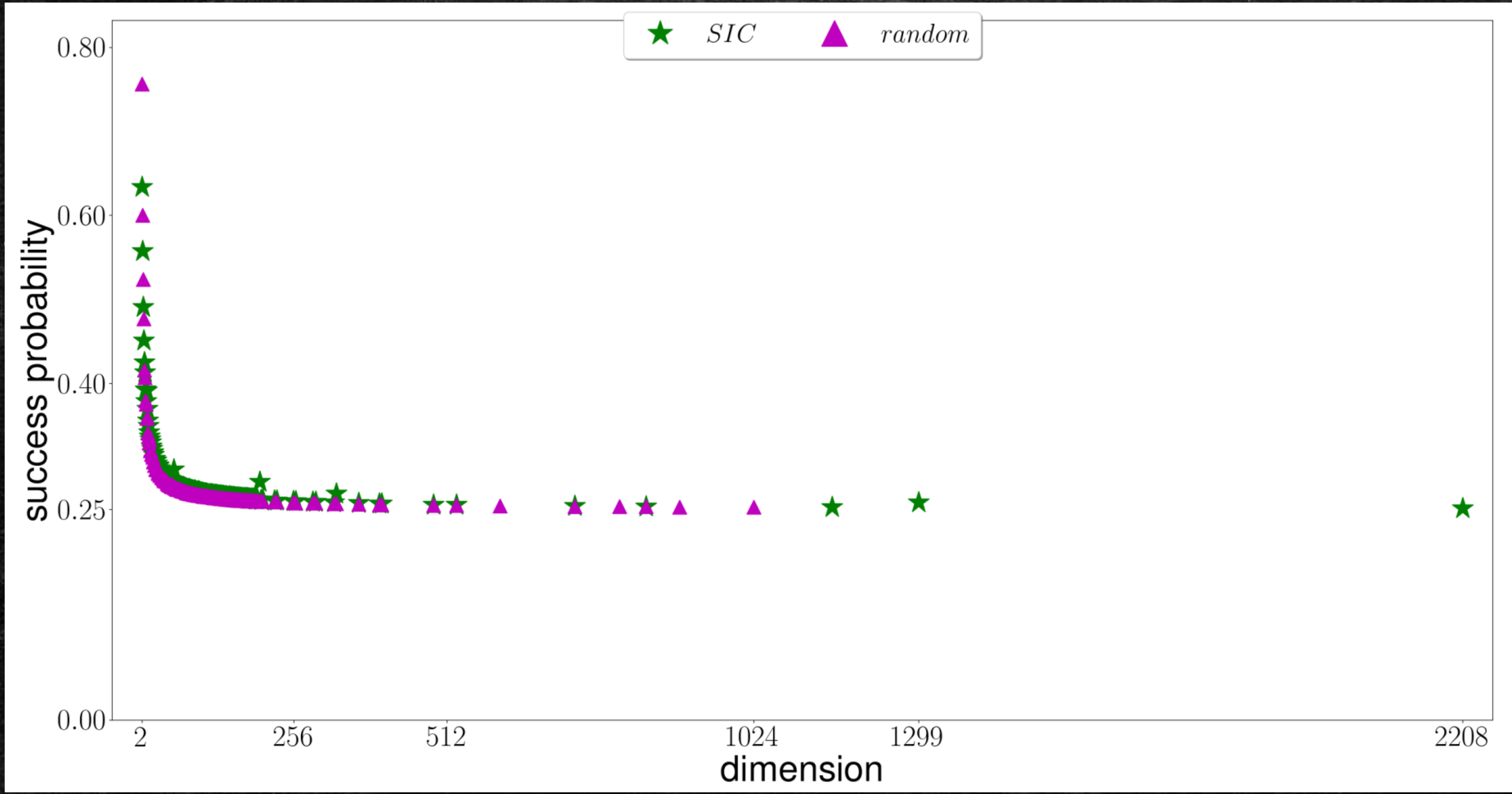

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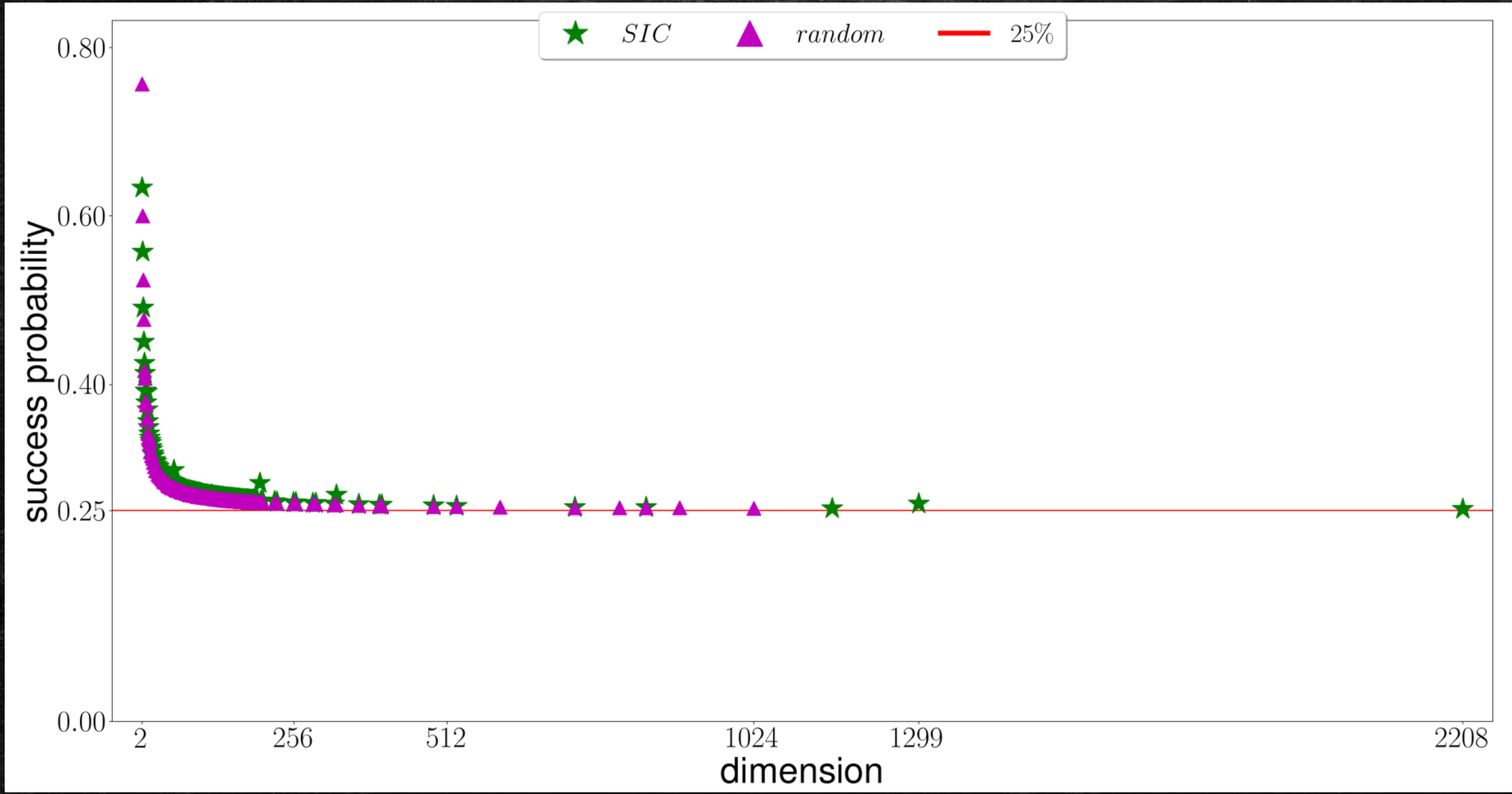

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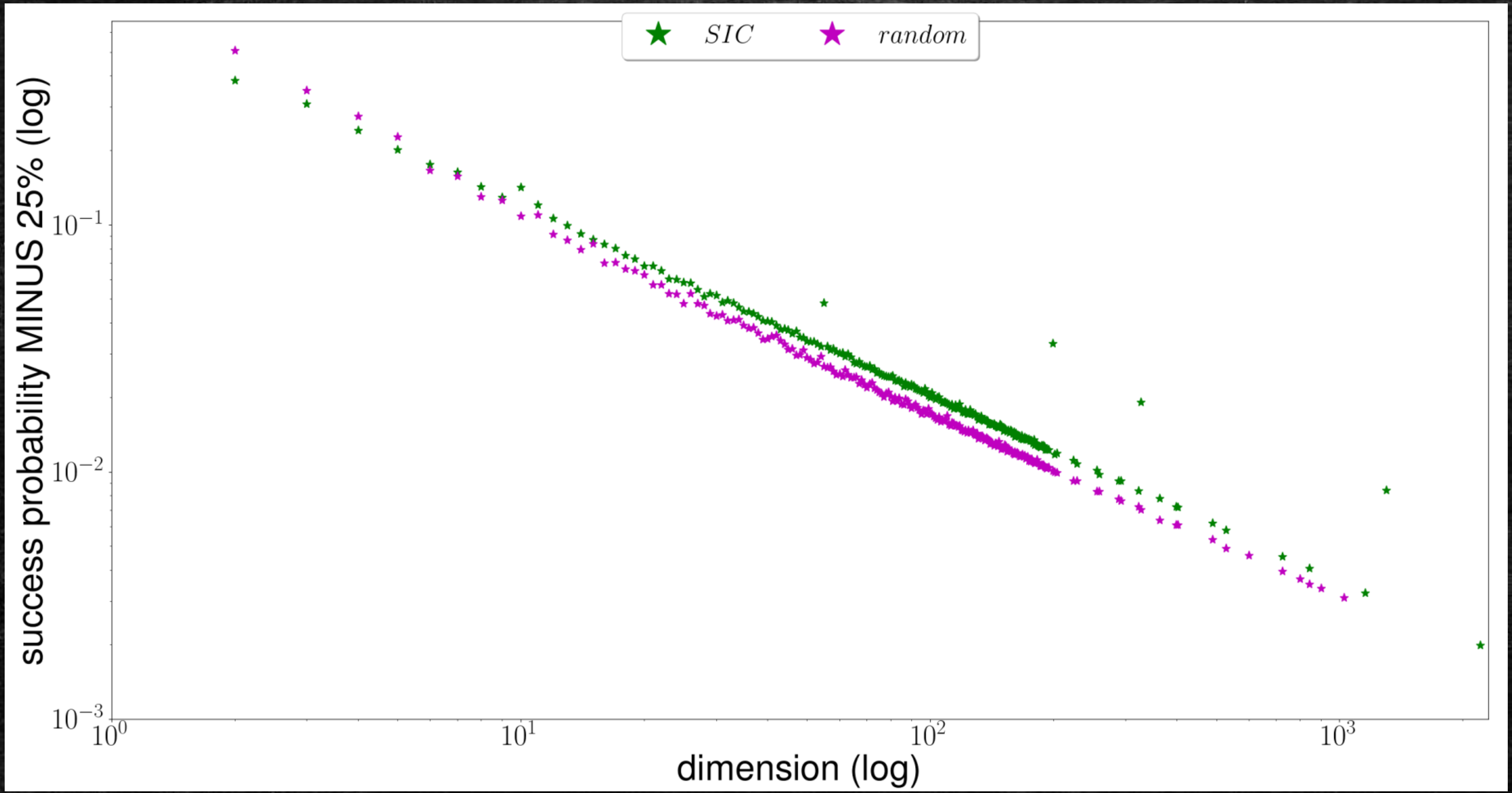

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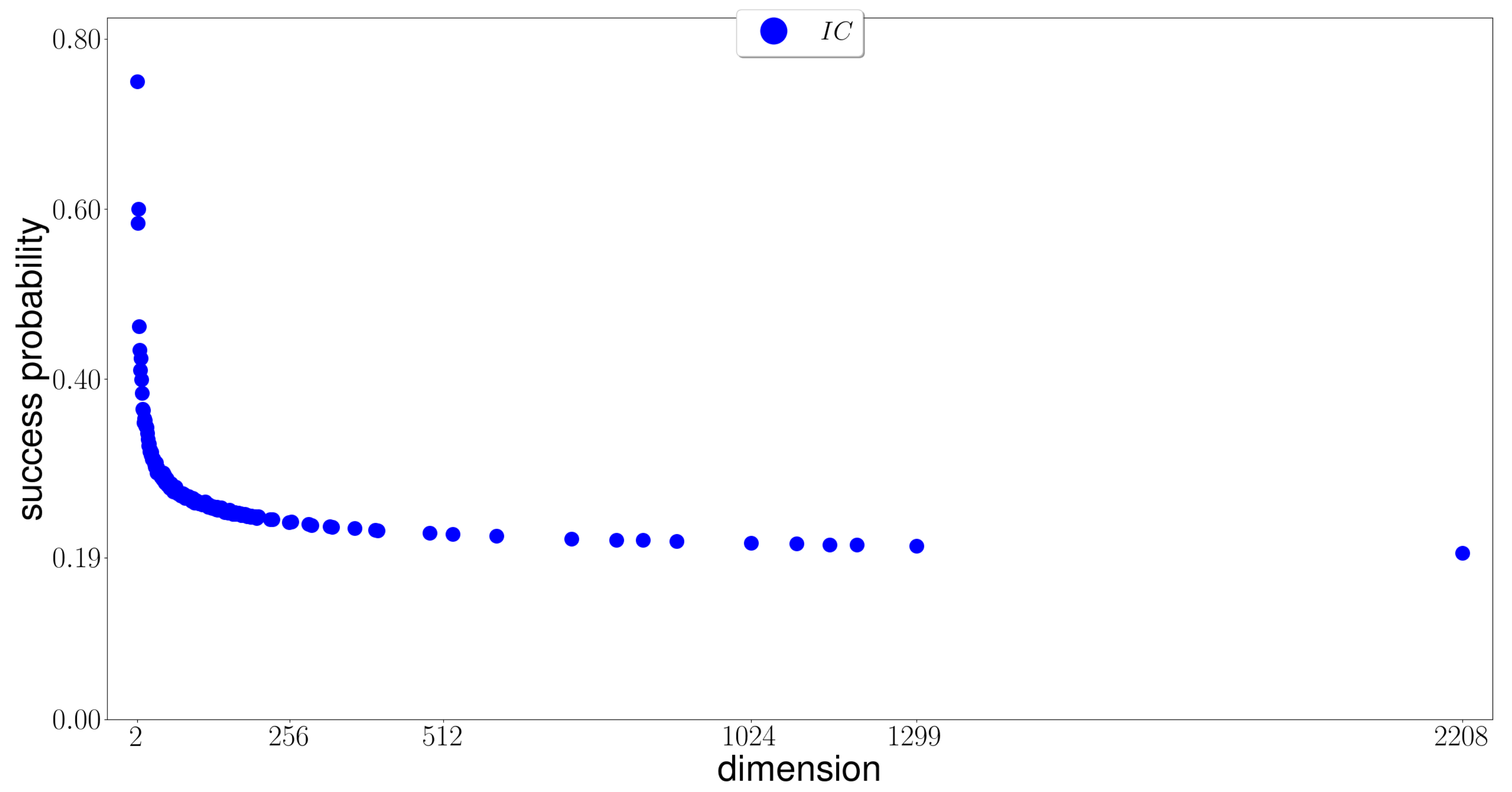

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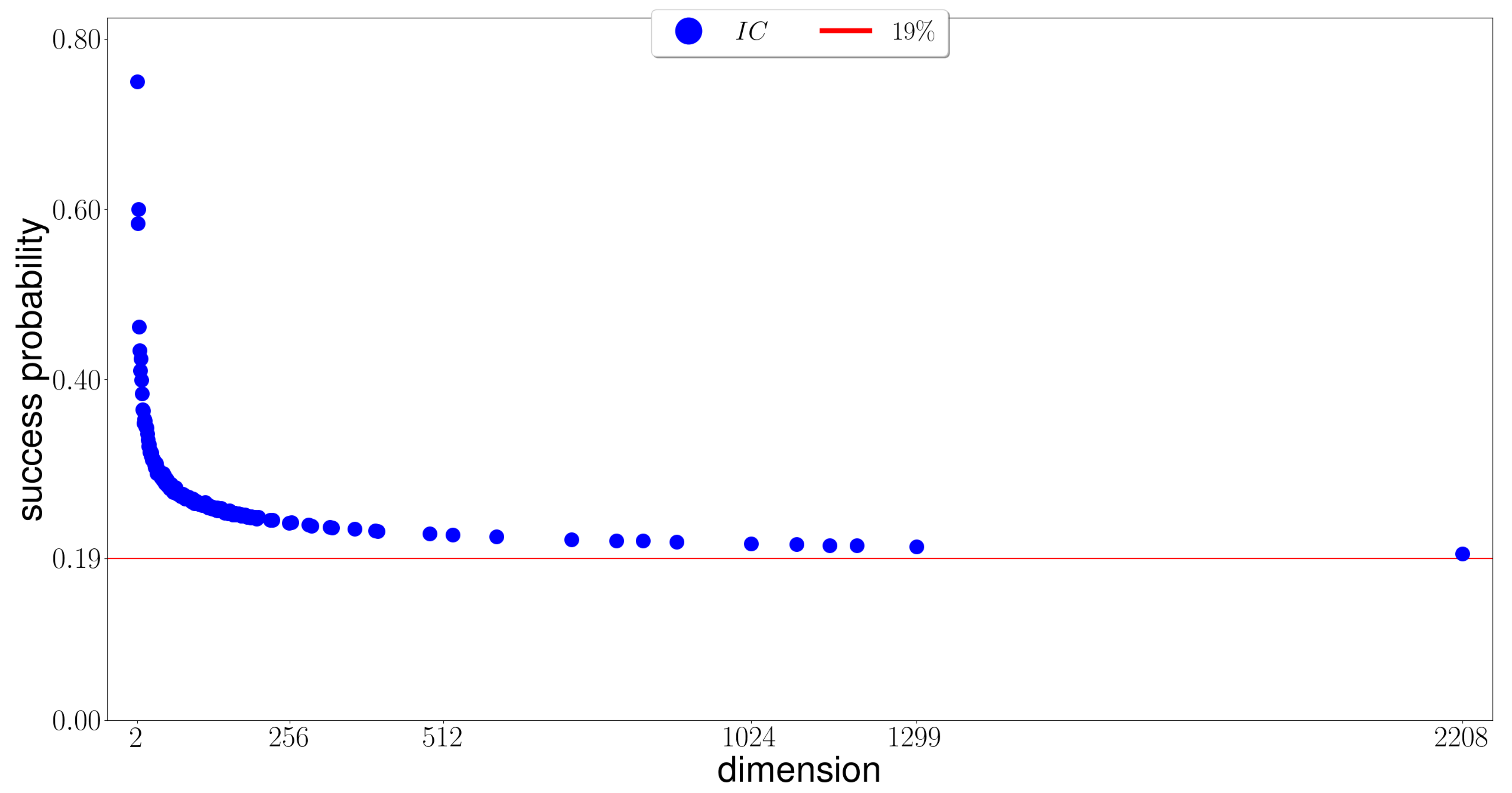

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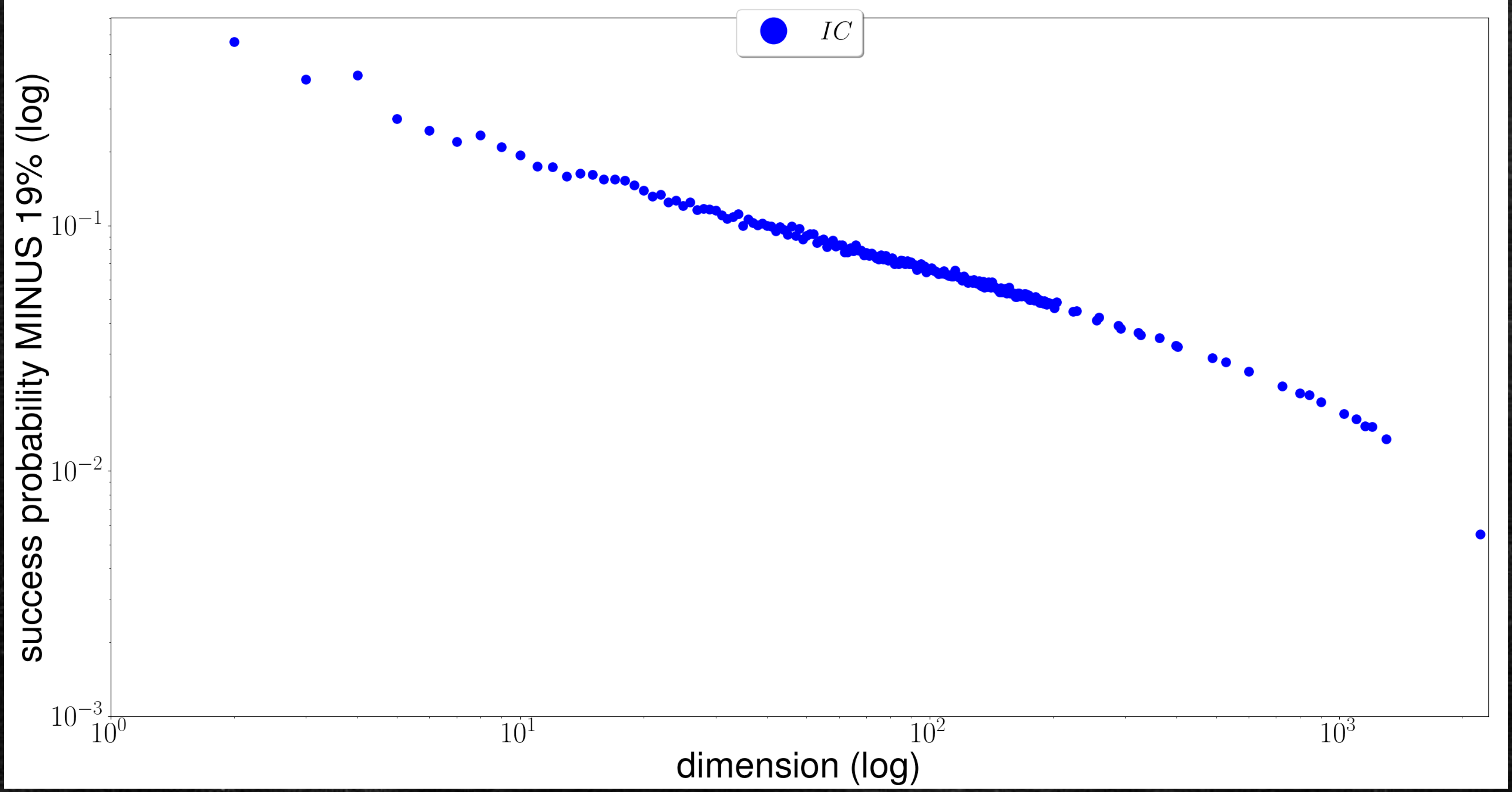

Y axis (logarithmic):
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X axis (logarithmic):
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Also have results for general n and m.

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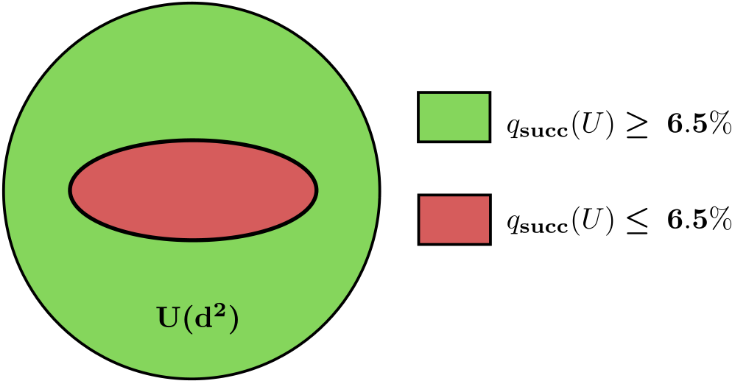

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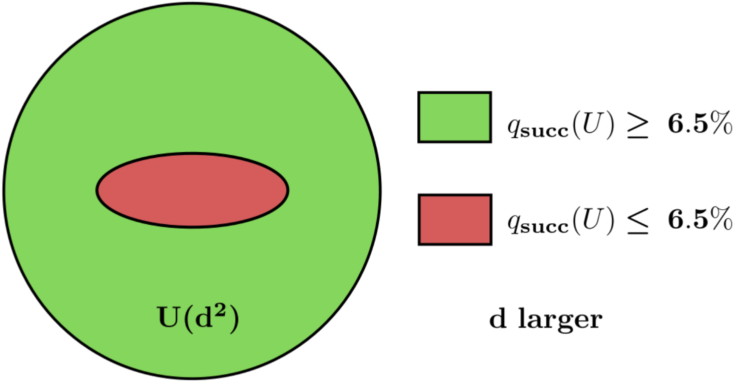

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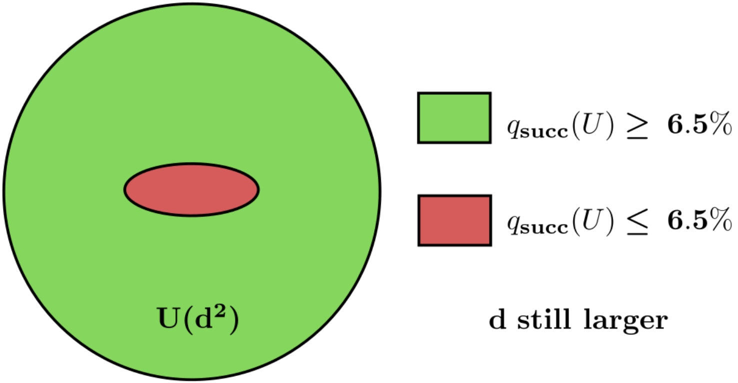

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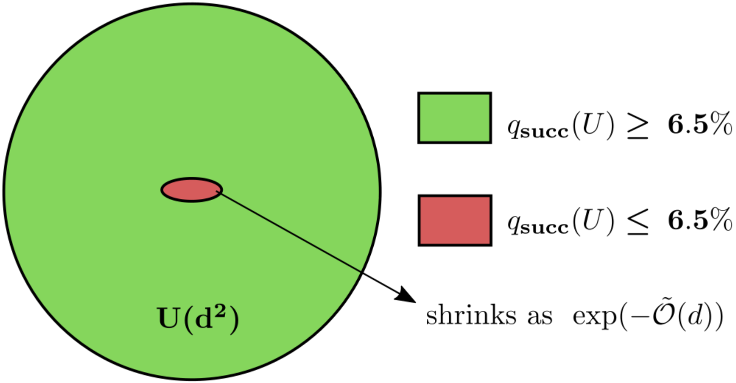

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Target POVM: Haar-random, $\mathbf{M} = (M_1, \cdots, M_{d^2})$

Noise model[2]: $M_i \longrightarrow \eta M_i + (1-\eta)\dfrac{\mathbb{1}}{d^2}$

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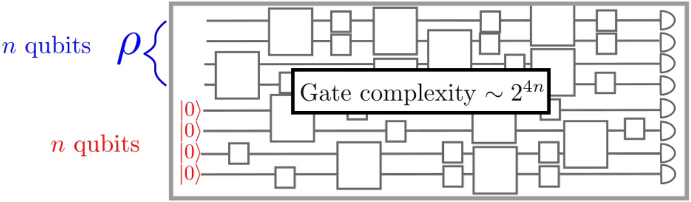

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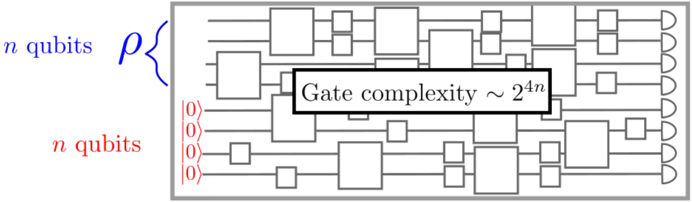

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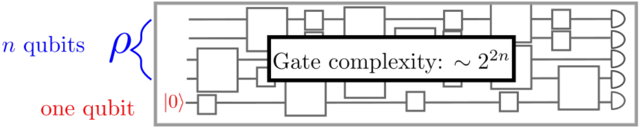

Our scheme: $\eta_{ps} \sim \exp(-\exp(2n))$

Our scheme's much better!

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- Noise compounding in circuits significantly less in our scheme, than Naimark dilation method.

Scope of our results

Scope in experiment

- Only a single ancillary qubit used.
- $\mathrm{q}_{\mathrm{succ}}$ is very high, for arbitrary dimension.

Scope in theory: applications and future directions

- Resource theory of measurements[a]
  - Simulating POVMs using restricted classes of POVMs,
  - Significance of $\mathrm{q}_{\mathrm{succ}}$: related to entanglement cost of measurement, visibility, etc?
- Non-locality: for e.g. randomness generation, local models .
- Simulation of random circuits with other circuits.

[a] Quantum 3 133 (2019)