Extended abstract: The axiomatic and the operational approach to resource theories of magic do not coincide

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INTRODUCTION

Stabiliser operations – Clifford gates, Pauli measurements and classical control – occupy a prominent role in the theory of fault-tolerant quantum computing. They can be efficiently simulated on a classical computer, a result which is known as the *Gottesman-Knill* theorem. However, an additional supply of *magic states* is enough to promote them to a universal, fault-tolerant model for quantum computing.

During recent years, there has been an increasing interest in developing a resource theory of quantum computing that allows for a precise quantification of *magic*. First resource theories were developed for the somewhat simpler case of odd-dimensional systems, based on a phase-space representation via Wigner functions. There, the total negativity in the Wigner function of a state is a *resource monotone* called *mana*, and non-zero mana is a necessary condition for a quantum speed-up [1–7]. In the more relevant case of qubits, this theory breaks down, which has led to a number of parallel developments [8–15]. A common element is that the finite set of stabiliser states, or more generally their convex hull, is taken as the set of free states. Since stabiliser operations preserve this *stabiliser polytope*, they are considered free operations in this theory and any monotones should be non-increasing under those. A number of such *magic monotones* have been studied and their values linked to the runtime of classical simulation algorithms [12, 16–18]. In this sense, the degree of magic present in a quantum circuit does seem to correlate with the quantum advantages it confers – thus validating the premise of the approach.

THE PROBLEM

The set of stabilizer operations (SO) are defined in terms of concrete actions ("prepare a stabilizer state, perform a Clifford unitary, make a measurement") and thus represent an *operational* approach to defining free transformations in a resource theory of magic. It is often fruitful to start from an *axiomatic* point of view, by defining the set of free transformations as those physical maps that preserve the set of free states. This approach has been introduced recently by Seddon and Campbell [10]. They suggest to refer to a linear map as *completely stabiliser-preserving* (CSP) if it preserves the stabiliser polytope, even when acting on parts of an entangled system. It has been shown that the magic monotones mentioned above are also non-increasing under CSP maps [12].

A natural and pressing question is therefore whether the two approaches coincide – i.e. whether SO = CSP, or whether there are CSP maps that cannot be realised as stabiliser operations [10].

To build an intuition for the question, consider the analogous problem in entanglement theory, where the free resources are the separable states. The axiomatically defined free transformations are the *separable maps* – completely positive maps that preserve the set of separable states. The operationally defined free transformations are those that can be realised by local operations and

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classical communication (LOCC). It is known that the set of separable maps is strictly larger than the set of LOCC [19, 20] – a fact that leads e.g. to a notable gap in the success probability of quantum state discrimination [21, 22] and entanglement conversion [23] between the two classes.

In this work, we show that – also in resource theories of magic – the axiomatic and the operational approaches lead to different classes, this is SO $\neq$ CSP.

MAIN RESULT

We show that any n-qubit CSP channel is of the form

$$\mathcal{E}(\rho) = \sum_{i=1}^{r} \frac{2^n \lambda_i}{\operatorname{rank} P_i} U_i P_i \rho P_i U_i^\dagger, \qquad \text{s.t.} \qquad \mathbb{1} = \sum_{i=1}^{r} \frac{2^n \lambda_i}{\operatorname{rank} P_i} P_i. \tag{1}$$

Here, $\lambda_i \geq 0$ are convex coefficients, i. e. $\sum_i \lambda_i = 1$, U_i are n-qubit Clifford unitaries, and P_i are stabiliser codes. The right equation is exactly the trace-preservation condition $\mathbb{1} = \mathcal{E}^\dagger(\mathbb{1})$. It states that the rescaled projective parts $\tilde{P}_i := (2^n \lambda_i / \operatorname{rank} P_i) P_i$ form a positive operator-valued measure (POVM). In this context, the CSP channel $\mathcal{E}$ can be seen as the instrument associated with the stabiliser POVM $\{\tilde{P}_i\}$ combined with the application of Clifford unitaries U_i conditioned on outcome i. A priori, this allows for more general quantum channels than in the case of stabiliser operations where the POVM has to come from Pauli measurements, and thus the $\tilde{P}_i$ are mutually orthogonal. In particular, one could think of arranging overlapping codes with the right weights in non-trivial ways such that they yield the identity on Hilbert space. Indeed, an example of such a channel is the following:

$$\Lambda(\rho) := \frac{1}{2^n} \left(2^n H^{\otimes n} P \rho P H^{\otimes n} + 2 \sum_{z \in \mathbb{F}_2^n \setminus 0} P_z \rho P_z \right), \tag{2}$$

where H is the Hadamard gate, $P := |0\rangle\langle 0|^{\otimes n}$ and the sum is over all stabiliser code projectors of the form $P_z := \frac{1}{2}(\mathbb{1} - Z(z))$ with $Z(z) := \otimes_i Z^{z_i}$ the n-qubit Pauli-Z operator associated to the bitstring $z \in \mathbb{F}_2^n$.

In fact, the essence of our paper is to show the following:

Lemma 1. *The CSP channel Λ is not a stabiliser operation.*

This directly implies our main result:

Theorem 1. *The set of CSP channels is strictly larger than the set of stabiliser operations.*

IMPLICATIONS

Theorem 1 indicates that recently proposed stabiliser-based simulation techniques of CSP maps are strictly more powerful than Gottesman-Knill-like methods [12].

Furthermore, our result has direct applications to the problem of quantifying resources required in the magic state model for quantum computing. The axiomatic approach to free operations has the advantage that it is possible to directly apply results from general resource theory and obtain explicit bounds on e.g. state conversion and distillation rates [4, 12, 24–26]. For this case, it is also known that the theory is asymptotically reversible [15]. Here, it would be interesting to study whether tasks like magic state distillation can show a gap in the achievable rates between CSP channels and stabiliser operations. Again, this question is motivated from entanglement theory, where a significant separation between separable channels and LOCC operations for e.g. entanglement conversion is known [23].

TECHNIQUES AND AUXILIARY RESULTS

More formally, let us define the *stabiliser polytope* SP_n as the convex hull of n-qubit stabiliser state projectors and CSP_n as the set of n-qubit CSP channels. As shown in Ref. [10], the set CSP_n is a convex polytope which is isomorphic to $\mathrm{SP}_{2n} \cap \mathrm{TP}$ under the Choi-Jamiołkowski isomorphism $\mathcal{J}$. Here, TP is the affine subspace defined by the trace-preservation (TP) condition $\mathrm{Tr}_1 \mathcal{J}(\mathcal{E}) = \mathbb{1}/2^n$. We show that any bipartite $2n$-qubit stabiliser state is proportional to $UP \otimes \mathbb{1} \left|\phi^+\right\rangle$, where $\left|\phi^+\right\rangle = 2^{-n/2} \sum_x |xx\rangle$ is the standard maximally entangled state, U is a Clifford unitary and P is a stabiliser code projector. This directly leads to the form (1) for CSP channels.

The proof of Lemma 1 is based on two components. First, we give a proof that the channel Λ is extremal within the polytope of CSP channels. Since $\mathrm{SO} \subset \mathrm{CSP}$, it might be intuitively clear that an example separating those sets should be close to the boundary of CSP. Second, we prove a technical property of extremal stabiliser operations. Then, the extremality of Λ allows us to separate it from SO by showing that Λ does not possess this property.

This separating property is based on a careful analysis of extremal stabiliser operations. Namely, we show that any extremal stabiliser operation either contains a non-trivial Pauli operator in its kernel or can be written as a partial Clifford isometry.

The proof that Λ is an extreme point of CSP combines the special structure of bipartite stabiliser states with techniques from convex geometry. The choice of stabiliser POVM defining Λ allows us to construct a special *pyramidal face* of the CSP polytope with apex (or "tip") Λ which proves that Λ is an extreme point of CSP.

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[1] E. F. Galvão, Discrete wigner functions and quantum computational speedup, Physical Review A **71**, 042302 (2005).

[2] D. Gross, Non-negative wigner functions in prime dimensions, Applied Physics B **86**, 367 (2007).

[3] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, Negative quasi-probability as a resource for quantum computation, New Journal of Physics **14**, 113011 (2012).

[4] V. Veitch, S. A. H. Mousavian, D. Gottesman, and J. Emerson, The resource theory of stabilizer quantum computation, New Journal of Physics **16**, 013009 (2014).

[5] A. Mari and J. Eisert, Positive wigner functions render classical simulation of quantum computation efficient, Physical Review Letters **109**, 230503 (2012).

[6] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Contextuality supplies the 'magic' for quantum computation, Nature **510**, 351 (2014).

[7] N. Delfosse, C. Okay, J. Bermejo-Vega, D. E. Browne, and R. Raussendorf, Equivalence between contextuality and negativity of the wigner function for qudits, New Journal of Physics **19**, 123024 (2017).

[8] M. Howard and E. Campbell, Application of a resource theory for magic states to fault-tolerant quantum computing, Physical Review Letters **118**, 10.1103/PhysRevLett.118.090501 (2017).

[9] M. Heinrich and D. Gross, Robustness of magic and symmetries of the stabiliser polytope, Quantum **3**, 132 (2019).

[10] J. R. Seddon and E. T. Campbell, Quantifying magic for multi-qubit operations, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **475**, 20190251 (2019).

[11] R. Raussendorf, J. Bermejo-Vega, E. Tyhurst, C. Okay, and M. Zurel, Phase space simulation method for quantum computation with magic states on qubits, Phys. Rev. A 101, 012350 (2020) 10.1103/PhysRevA.101.012350 (2019).

[12] J. R. Seddon, B. Regula, H. Pashayan, Y. Ouyang, and E. T. Campbell, Quantifying quantum speedups: improved classical simulation from tighter magic monotones, arXiv:2002.06181 [quant-ph] (2020).

[13] M. Beverland, E. Campbell, M. Howard, and V. Kliuchnikov, Lower bounds on the non-clifford resources for quantum computations, Quantum Science and Technology **5**, 035009 (2020).

[14] A. Heimendahl, F. Montealegre-Mora, F. Vallentin, and D. Gross, Stabilizer extent is not multiplicative, arXiv:2007.04363 [quant-ph] (2020).

[15] Z.-W. Liu and A. Winter, Many-body quantum magic, arXiv:2010.13817 [cond-mat, physics:quant-ph] (2020), 2010.13817.

[16] H. Pashayan, J. J. Wallman, and S. D. Bartlett, Estimating outcome probabilities of quantum circuits using quasiprobabilities, Physical Review Letters **115**, 070501 (2015).

[17] S. Bravyi and D. Gosset, Improved classical simulation of quantum circuits dominated by clifford gates, Physical Review Letters **116**, 250501 (2016).

[18] S. Bravyi, D. Browne, P. Calpin, E. Campbell, D. Gosset, and M. Howard, Simulation of quantum circuits by low-rank stabilizer decompositions, Quantum **3**, 181 (2019).

[19] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Quantum nonlocality without entanglement, Physical Review A **59**, 1070 (1999).

[20] E. Chitambar, D. Leung, L. Mančinska, M. Ozols, and A. Winter, Everything you always wanted to know about LOCC (but were afraid to ask), Communications in Mathematical Physics **328**, 303 (2014).

[21] M. Koashi, F. Takenaga, T. Yamamoto, and N. Imoto, Quantum nonlocality without entanglement in a pair of qubits, arXiv:0709.3196 [quant-ph] (2007).

[22] R. Duan, Y. Feng, Y. Xin, and M. Ying, Distinguishability of Quantum States by Separable Operations, IEEE Transactions on Information Theory **55**, 1320 (2009).

[23] E. Chitambar, W. Cui, and H.-K. Lo, Increasing Entanglement Monotones by Separable Operations, Physical Review Letters **108**, 240504 (2012).

[24] Z.-W. Liu, One-shot operational quantum resource theory, Physical Review Letters **123**, 10.1103/PhysRevLett.123.020401 (2019).

[25] K. Fang and Z.-W. Liu, No-go theorems for quantum resource purification, Physical Review Letters **125**, 060405 (2020).

[26] X. Wang, M. M. Wilde, and Y. Su, Efficiently computable bounds for magic state distillation, Physical Review Letters **124**, 090505 (2020).