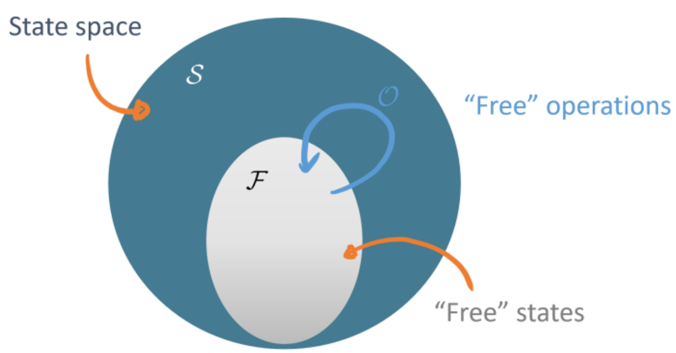
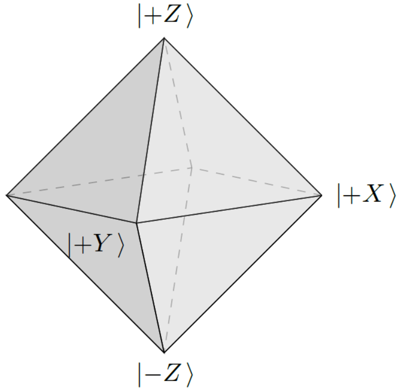
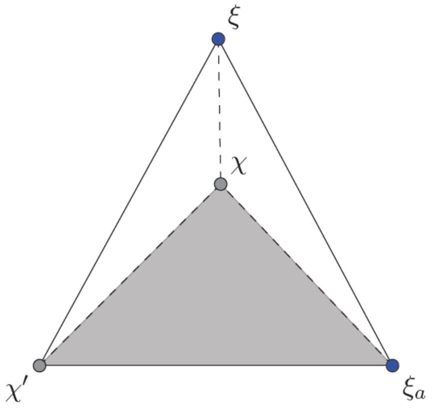
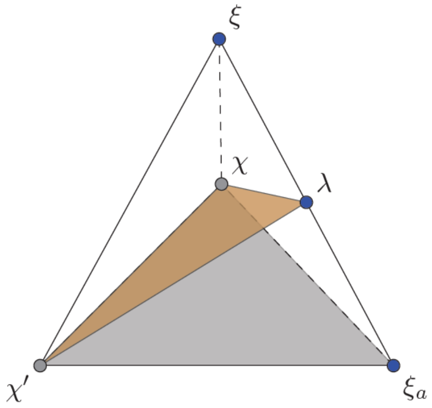








The axiomatic and operational approaches to resource theories of magic do not coincide

Markus Heinrich (Heinrich Heine University Düsseldorf & University of Cologne)

QIP 2021| 5 February 2021

The axiomatic and operational approaches to resource theories of magic do not coincide

arXiv:2011.11651

Arne Heimendahl
(math department)

David Gross

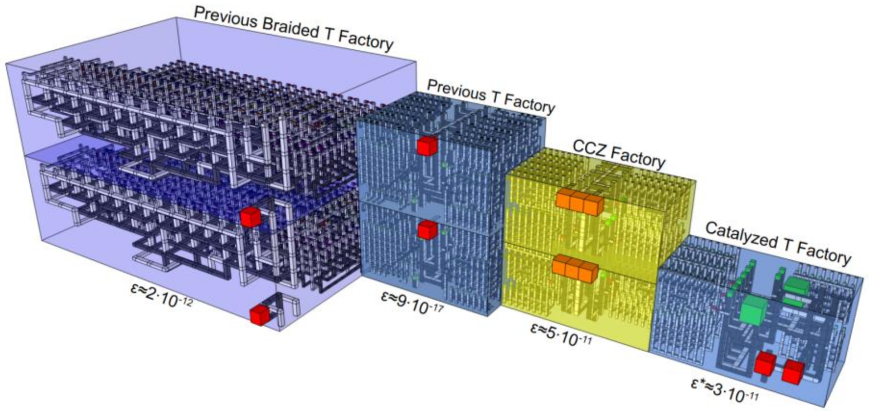
Markus Heinrich

University of Cologne

Heinrich Heine University Düsseldorf

Motivation

"Magic state factory"



[Gidney and Fowler 2019]

Magic state model = Model of fault-tolerant universal QC

Problem: Huge overhead

Resource theory of magic ⟷ (quantifying "magic") ⟷ Classical simulation

↓

Distillation rates

Achievable via

→ "Stabiliser operations" ?

→ "Stabiliser-preserving channels" ?

Introduction

Stabiliser operations, Gottesman-Knill, and resources in quantum computing

Stabiliser operations and Gottesman-Knill

A *stabiliser operation* (SO) is a circuit consisting of

- Preparation and measurement in the computational basis
- Application of phase, Hadamard and controlled-NOT gates
- Classical randomness and control

"Clifford" circuit

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Gottesman-Knill theorem: SO are efficiently simulable on a classical computer

$\Longrightarrow$ Not "powerful" in a quantum computational sense

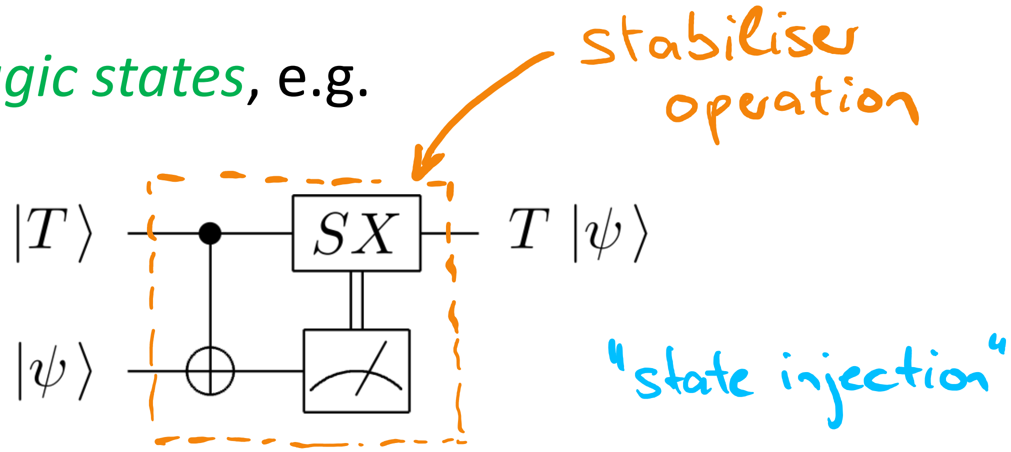
(otherwise very useful!)

Going universal: Magic states

Other *diagonal gates* can be applied using *magic states*, e.g.

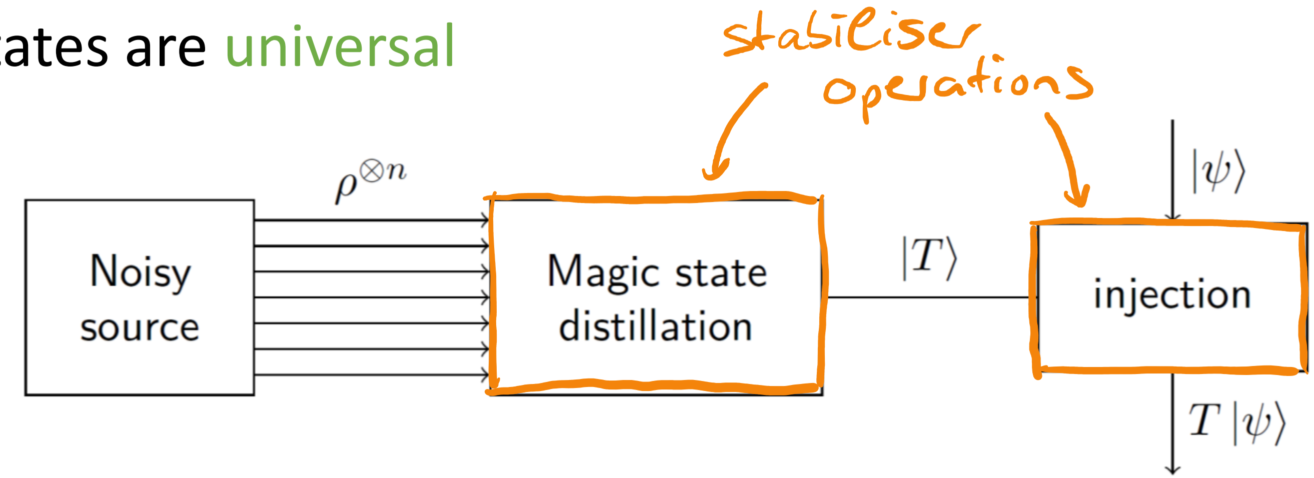
$$T = \mathrm{diag}(1, e^{i\pi/4})$$

$$\Rightarrow |T\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/4}|1\rangle\right)$$



Fact: Stabiliser operations + magic states are universal

Fact: Noisy states can be distilled

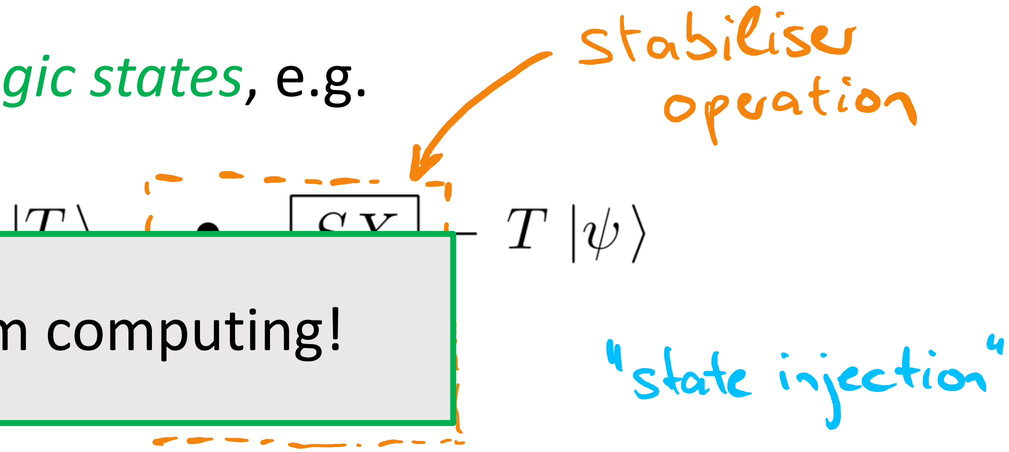


[Bravyi and Kitaev 2005]

Going universal: Magic states

Other *diagonal gates* can be applied using *magic states*, e.g.

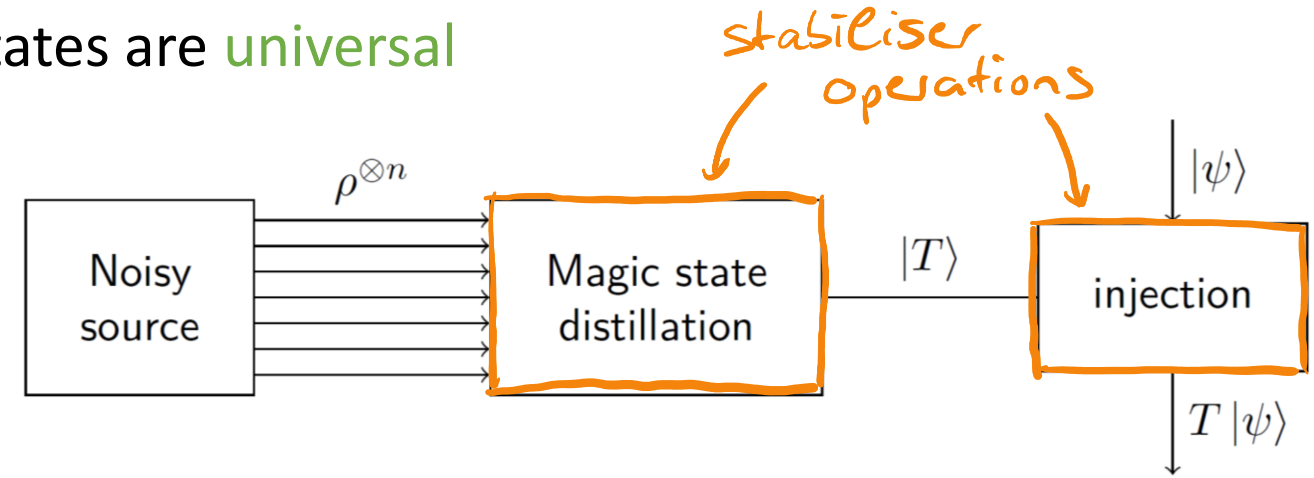
$T = \mathrm{diag}(1, e^{i\pi/4})$



Magic states are a resource for quantum computing!

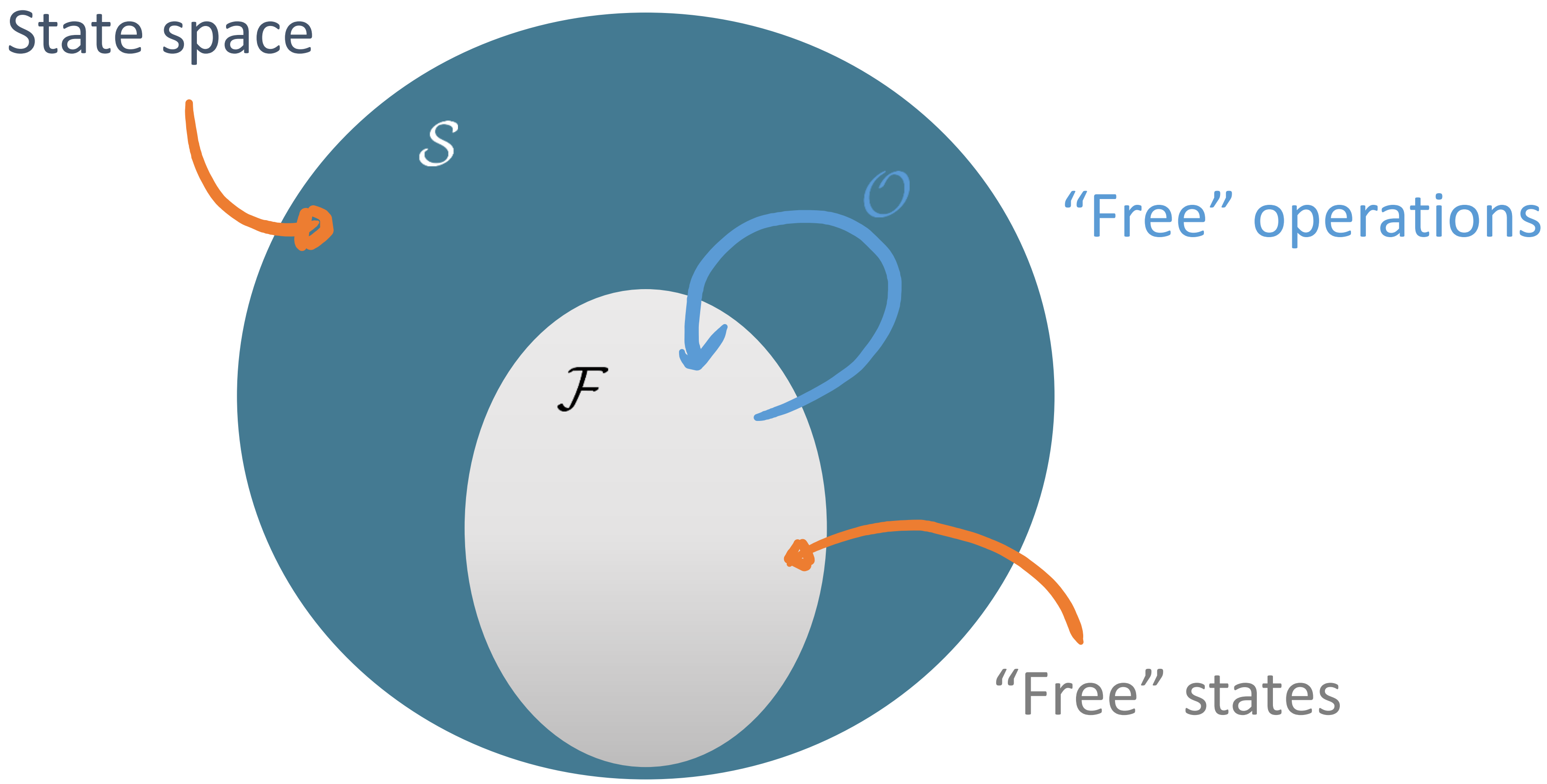
Fact: Stabiliser operations + magic states are universal

Fact: Noisy states can be distilled

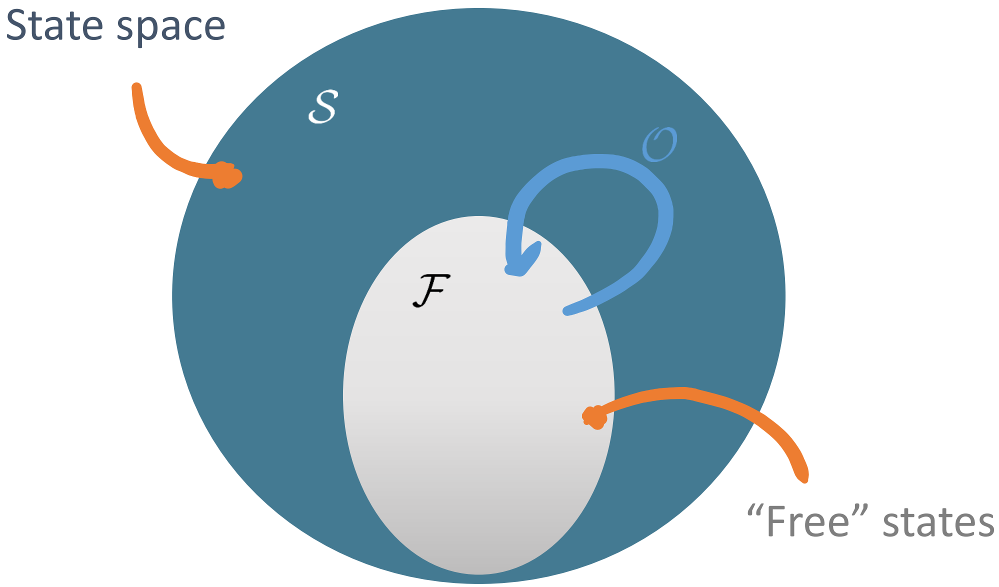


[Bravyi and Kitaev 2005]

Resource theories in a nutshell



Resource theories in a nutshell

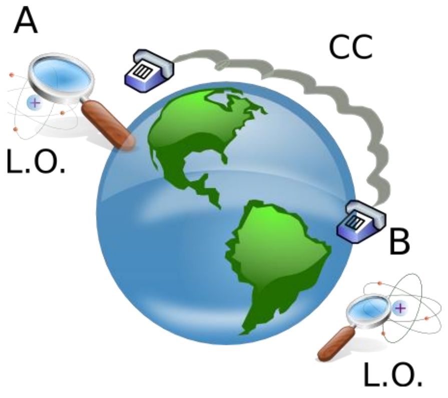


“Free” operations

Operational:
“Prepare this, apply that, measure those”

Axiomatic:
“Any channel which maps $\mathcal{F}$ to itself”

Example: Entanglement



Free states = separable states

Free operations:

Operational: *Local operations and classical communication* (LOCC)

Axiomatic: *Separable channels* (SEP)

"Classic" result:

$$\mathrm{LOCC}_1 \subsetneq \mathrm{LOCC}_k \subsetneq \mathrm{LOCC}_{k+1} \subsetneq \mathrm{LOCC} \subsetneq \overline{\mathrm{LOCC}} \subsetneq \mathrm{SEP}$$

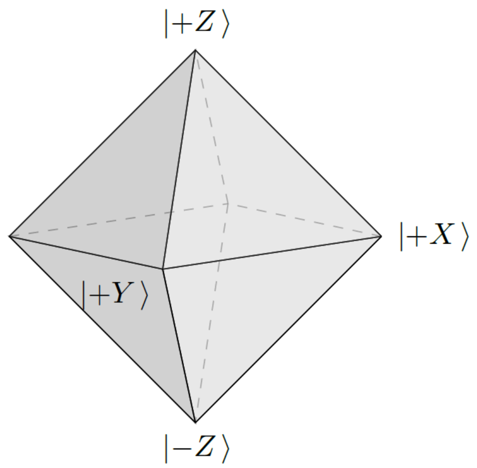
[Bennett et al. 1999; Chitambar et al. 2014]

“Magic” as a resource

Free states := states which can be prepared by SO = stabiliser polytope (for qubits)

Fact: This is exactly the convex hull of all stabiliser states, i.e.

$$\mathrm{SP} := \mathrm{conv}\left\{ U\,|0\rangle \;\middle|\; U \text{ is a Clifford unitary} \right\}$$



Free operations:

Operational: *stabiliser operations* (SO)

Axiomatic: *(compl.) stabiliser-preserving channels* (CSP): all channels which map SP to itself

[Ahmadi et al. 2018; Seddon and Campbell 2019; Seddon et al. 2020]

Spoiler alert

Not much is known about the SO and CSP classes … We show by explicit example:

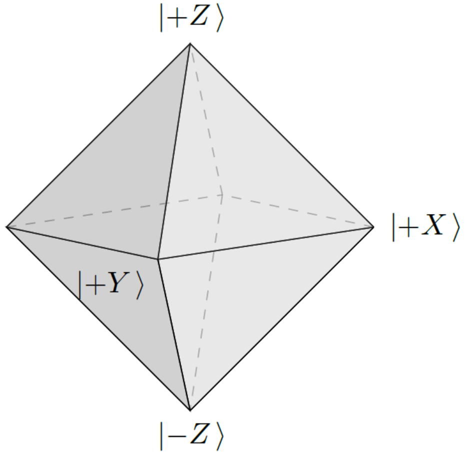
$$\mathrm{SO} \subsetneq \mathrm{CSP}$$

… and develop a characterization of CSP on the way

Completely stabiliser-preserving channels

characterisation, interpretation, and differentiation from stabiliser operations

CSP channels



A superoperator $\mathcal{E}$ is *completely stabiliser-preserving* iff it maps the stabiliser polytope to itself (even when applied to subsystems).

Lemma 1. *Any CSP map is completely positive and trace-preserving.*

e.g. $|\Phi^+\rangle = 2^{-n/2} \sum_x |xx\rangle$ is stabiliser state $\Rightarrow$ Choi state $\mathcal{J}(\mathcal{E}) := \mathcal{E} \otimes \mathrm{id}\, |\Phi^+\rangle \geq 0$

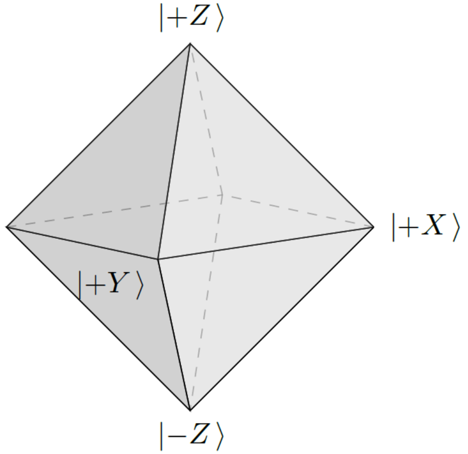
Lemma 2 (Lem. 4.2 in [13]). *A linear map $\mathcal{E} : L((\mathbb{C}^2)^{\otimes n}) \to L((\mathbb{C}^2)^{\otimes m})$ is CSP if and only if its Choi representation lies in the intersection of the stabiliser polytope with the affine space $\mathrm{TP}_{n,m}$:*

$$\mathcal{J}(\mathrm{CSP}_{n,m}) = \mathrm{SP}_{n,m} \cap \mathrm{TP}_{n,m}. \tag{11}$$

$\Rightarrow$ Write $\mathcal{J}(\mathcal{E})$ as convex combination of stabiliser states and impose TP

[Seddon and Campbell 2019; Heimendahl, MH, and Gross 2020]

CSP channels



Any bipartite 2n-qubit stabiliser state is of the form [Thm. 1, HHG]

$$|s\rangle = 2^{k/2} U P \otimes \mathbb{1} \left|\Phi^+\right\rangle$$

Where

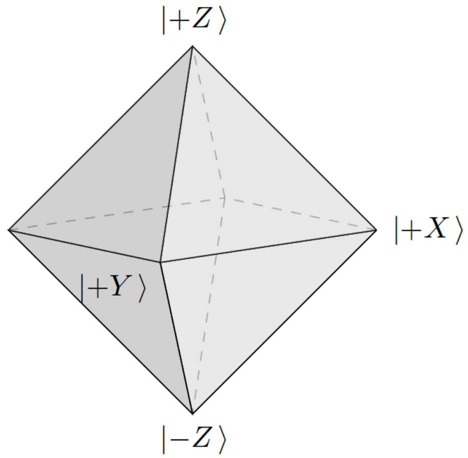
- U is a Clifford unitary
- P projects onto a subspace spanned by orthonormal stabiliser states (stabiliser code)

$\Rightarrow$ Write $\quad \mathcal{E}(\rho) = \sum_i \lambda_i 2^{k_i} U_i P_i \rho P_i U_i^\dagger \quad$ s.t. $\quad \mathbb{1} = \mathcal{E}^\dagger(\mathbb{1}) = \sum_i \lambda_i 2^{k_i} P_i$

Interpretation: Perform POVM $E_i := \lambda_i 2^{k_i} P_i$ followed by U_i conditioned on outcome i

[Heimendahl, MH, and Gross 2020]

CSP examples



$$\mathcal{E}(\rho) = \sum_i \lambda_i U_i \rho U_i^\dagger$$

$$\mathcal{E}(\rho) = \sum_x U_x \, |x\rangle\langle x| \, \rho \, |x\rangle\langle x| \, U_x^\dagger$$

$$\mathcal{E}(\rho) = \sum_i U_i \, |s_i\rangle\langle s_i| \, \rho \, |s_i\rangle\langle s_i| \, U_i^\dagger$$

Where $|s_i\rangle$ is an orthonormal stabiliser basis

These are all stabiliser operations! We need non-orthogonal projectors!

[Heimendahl, MH, and Gross 2020]

CSP != SO

Central theorem: The following is CSP and not SO for $n \geq 2$

$$\Lambda(\rho) = \rho_{00} \left|+\right\rangle\!\left\langle+\right| + \sum_{x\neq 0} \rho_{xx} \left|x\right\rangle\!\left\langle x\right| + \frac{1}{2} \sum_{x\neq y\neq 0} \rho_{xy} \left|x\right\rangle\!\left\langle y\right|$$

1. Projective measurement of $\{ \left|0\right\rangle\!\left\langle 0\right|, \mathbb{1} - \left|0\right\rangle\!\left\langle 0\right| \}$
2. Dephasing in the computational basis with probability 1/2
3. Conditioned on "0" perform global Hadamard

not CSP by itself

Moreover: CSP = SO for $n = 1$

"possible measurements become orthogonal"

[Heimendahl, MH, and Gross 2020]

CSP != SO

Central theorem: The following is CSP and not SO for $n \geq 2$

$$\Lambda(\rho) = \rho_{00} \, |+\rangle\langle+| + \sum_{x \neq 0} \rho_{xx} \, |x\rangle\langle x| + \frac{1}{2} \sum_{x \neq y \neq 0} \rho_{xy} \, |x\rangle\langle y|$$

Proof strategy: Show that Λ is …

1. a CSP channel (okay ✓)
2. extremal within the CSP set (totally not obvious)
3. not a stabiliser operation (not obvious, extremality needed)

[Heimendahl, MH, and Gross 2020]

Technical sneak peak

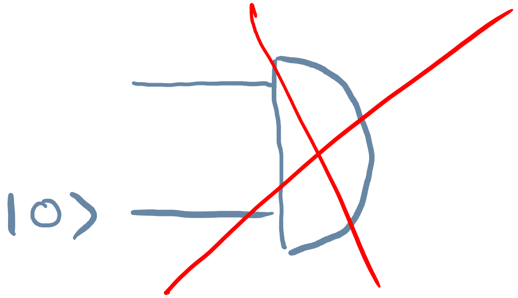
extremality, convex geometry and all that

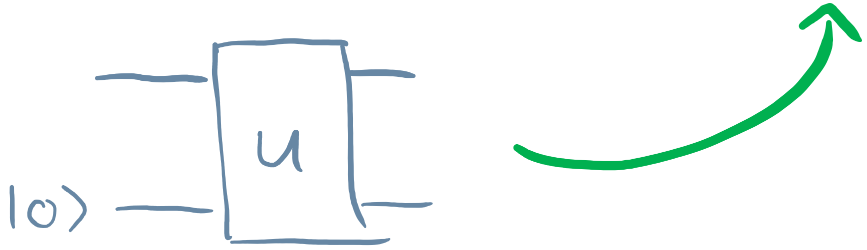
Invariance property of SO

How do you separate SO from CSP ?

$\mathcal{E} =$  $\Rightarrow \mathcal{E}(ZsZ) = \mathcal{E}(s)$

Claim: extremal stabiliser operations have a Pauli invariance (except for Clifford dilations)





Careful argumentation shows that such a SO cannot be extremal

Lambda is not SO

Separate SO from CSP using:

Theorem 3 (Pauli invariance of extremal stabiliser operations). *Let $\mathcal{O} \in \mathrm{SO}_{n,m}$ be an extremal stabiliser operation. Then, at least one of the following is true:*

(i) *There is a $x \in \mathbb{F}_2^{2n} \setminus 0$ such that $\mathcal{O} = \mathcal{O} \circ \mathrm{Ad}(w(x))$.*

(ii) *$\mathcal{O}$ has a Clifford dilation.*

↳ non-trivial Pauli operator

Claim follows from the observation that Λ

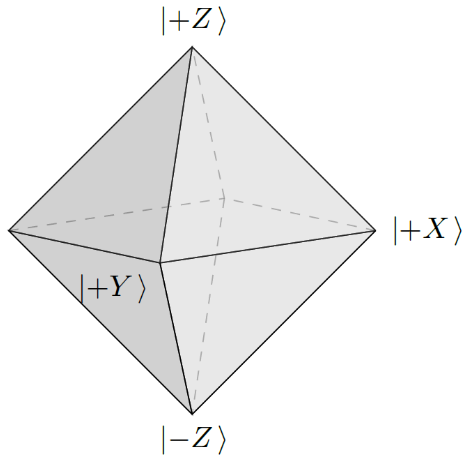
1. is extremal
2. is not a Clifford dilation
3. does not have a Pauli invariance

The Lambda channel

Credits to Arne

How do you come up with … ?

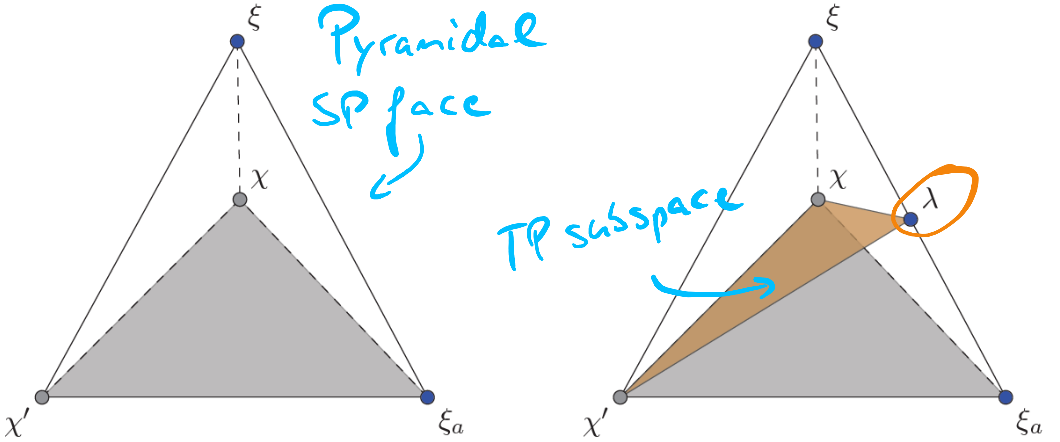
$$\Lambda(\rho) = \rho_{00} \, |+\rangle\langle+| + \sum_{x\neq 0} \rho_{xx} \, |x\rangle\langle x| + \frac{1}{2} \sum_{x\neq y\neq 0} \rho_{xy} \, |x\rangle\langle y|$$



… we were looking for extremal CSP channels

1. “likely” to not be SO
2. needed to use Pauli invariance of SO
3. Interesting by themselves, e.g. for simulation questions
4. Because we like math :)

Our channel is the result of a vertex construction



[Heimendahl, MH, and Gross 2020]

Summary

Resource-theoretic perspective on quantum computing is still in its infancy

Our contribution:

- Studying the CSP class of operations
- Showing that CSP is strictly larger than SO
- Initiating the study of extremal CSP channels

Outlook:

- Implications on magic state distillation / resource conversion ? *Gaps*?
- Simulation of CSP seems possible (“*beyond Gottesman-Knill*”) [Seddon et al. 2020]

However, details have to be filled in …

Thank you for your attention!

Live long and prosper!

Lambda channel with stabiliser codes

Central theorem: The following is CSP and not SO for $n \geq 2$

$$\Lambda(\rho) = \rho_{00} \, |+\rangle\langle+| + \sum_{x \neq 0} \rho_{xx} \, |x\rangle\langle x| + \frac{1}{2} \sum_{x \neq y \neq 0} \rho_{xy} \, |x\rangle\langle y|$$

This can be written using the stabiliser codes $P_z := \frac{1}{2}(\mathbb{1} - Z(z))$

$$\Lambda(\rho) = H^{\otimes n} \, |0\rangle\langle 0| \, \rho \, |0\rangle\langle 0| \, H^{\otimes n} + \frac{1}{2^{n-1}} \sum_{z \in \mathbb{F}_2^n \setminus 0} P_z \rho P_z$$

One can check that

$$|0\rangle\langle 0| + \frac{1}{2^{n-1}} \sum_{z \in \mathbb{F}_2^n \setminus 0} P_z = \mathbb{1}$$

[Heimendahl, MH, and Gross 2020]

Magic state distillation

If single-qubit conversion $\rho^{\otimes k} \to \sigma^{\otimes m}$ is possible via CSP:

$$GR(\rho^{\otimes k}) \geq GR(\sigma^{\otimes m}) \quad \Rightarrow \quad \frac{m}{k} \leq \frac{\log GR(\rho)}{\log GR(\sigma)}$$

Upper bound is achievable if conversion is *reversible*

[Liu-Winter 2020]: Resource theory of magic is *asymptotically reversible* (w.r.t CSP channels)

➡ Gap is possible! (maybe along the lines of [Chitambar et al. 2012])

Magic monotones

There are a number of magic monotones w.r.t. to CSP

- (Free) robustness of magic
- Generalised robustness
- dyadic negativity
- mixed-state extent

} mixed state extensions of stabiliser extent for pure states

(Some of) these monotones can be tied to runtimes of appropriate simulation algorithms