Entangleability of cones

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Introduction. Quantum entanglement is usually thought of as a genuine signature of quantum mechanics, deeply rooted within its formalism and intimately connected with the existence of superpositions. Such a connection between a local phenomenon (superposition) and a global one (entanglement) seems however a mere consequence of the peculiar mathematical form of quantum theory. In our work we show that this is not the case, and that *any* pair of non-classical local theories satisfying some general axioms has the potential to give rise to entanglement at the level of bipartite systems. This entails that non-classicality and entangleability are profoundly connected notions.

To model physical systems subjected to only the most basic operational axioms we employ the well-known mathematical machinery of general probabilistic theories (GPTs) [5, 8]. As predicted by Ludwig's embedding theorem [5–7], this broad framework encompasses all physical theories whose operational power obeys minimal requirements, such as the capability of predicting the probabilities associated with measurement outcomes. It contains classical probability and quantum mechanics as special cases, but includes also a wealth of alternative theories that may or may not be relevant for future physics. A few physical principles, such as the necessity to account for states and measurements, lead to a clean mathematical representation of GPTs. A GPT can be described by a triple (V, C, u), where V is a finite-dimensional[1] real vector space, $C \subset V$ is a convex cone and the unit u is a linear form on V that is positive on $C \setminus \{0\}$. As mentioned, the simplest examples of GPTs are classical theories, defined by the fact that the cone C is simplex-based, i.e. it is of the form $C = \mathrm{cone}\{v_1, \ldots, v_d\}$, where $\{v_1, \ldots, v_d\}$ is a basis of the d-dimensional vector space V.

Besides simple nondegeneracy conditions on C, we also require our model to obey the so-called *no-restriction hypothesis* [4], which implies that any functional taking on values between 0 and 1 on all states corresponds to a physically observable effect. Our main motivation for making this non-trivial assumption is twofold: on the one hand, it is satisfied by both classical and quantum theory, while on the other it leads to a neat and elegant mathematical model.

Entangleability. Entanglement arises when one tries to model multiple systems. Namely, given two physical systems represented in the GPT formalism by triples $A = (V_1, C_1, u_1)$ and $B = (V_2, C_2, u_2)$, how to represent also the joint system as a GPT $AB = (V_{12}, C_{12}, u_{12})$? A natural assumption that is often made in this context is the *local tomography principle*: bipartite states are uniquely determined by their statistics under local measurements. Note that both classical probability theory and quantum mechanics satisfy the local tomography principle. The naturalness of this assumption in this context is bolstered by the fact that dropping it trivialises the problem from the mathematical perspective [3].

Thanks to local tomography, one derives that the composite system is of tensor product form: it must be that $V_{12} = V_1 \otimes V_2$ and $u_{12} = u_1 \otimes u_2$. However, there appears to be no indisputable physical axiom that allows us to specify the cone C_{12}, beyond the inclusions

$$C_1 \odot C_2 \subseteq C_{12} \subseteq C_1 \circledast C_2, \tag{1}$$

[1] This assumption is made for technical reasons.

where $C_1 \odot C_2$ and $C_1 \circledast C_2$ are the *minimal and maximal tensor products*:

$$C_1 \odot C_2 \coloneqq \operatorname{conv}\{x \otimes y : \ x \in C_1,\, y \in C_2\}\,, \tag{2}$$
$$C_1 \circledast C_2 \coloneqq \{z \in V_1 \otimes V_2 : \ (f \otimes g)(z) \geqslant 0 \ \forall\, f \in C_1^*,\, g \in C_2^*\}\,. \tag{3}$$

Here, C^* denotes the cone dual to a cone C.

Consider a pair of GPTs $A = (V_1, C_1, u_1)$ and $B = (V_2, C_2, u_2)$. Motivated by the analogy with the quantum concept, we call the pair (A, B) or the pair (C_1, C_2) *entangleable* if $C_1 \odot C_2 \neq C_1 \circledast C_2$. Indeed, such a pair must exhibit some kind of of entanglement, either at the level of states or at the level of measurements. Our main question is then to decide:

Which pairs of GPTs are entangleable?

Main result and discussion. While two copies of the quantum theory are obviously entangleable, it can be shown that if either A or B is classical, the pair (A, B) is *not* entangleable. Our main result states that classicality of local theories is not only necessary but also sufficient to ensure the emergence of entanglement at the level of bipartite systems.

Theorem. *If neither C_1 nor C_2 is classical, then the pair (C_1, C_2) is entangleable.*

The above result demonstrates that there is a profound connection between the notion of non-classicality and that of entanglement, and that such a connection goes well beyond the contingent mathematical details of quantum theory. This is remarkable especially because the former concept is essentially local, while the latter has to do with bipartite systems. The fact that the two are inextricably related — under the only assumptions of no-restriction and local tomography — unifies them in a conceptually pleasing way.

The proof of our main result rests on few clear and intuitive ideas, yet it is mathematically quite involved. As it turns out, the question we answer was raised by Barker already in the 1970s [1, 2], with an entirely different motivation that had nothing to do with physics or information theory, but was on the contrary essentially algebraic. Prior to our work, notable progress on the problem had been made by Namioka and Phelps [9], who proved a weaker version of the above result: if C_1 is not classical, then there is a cone C_2 such that (C_1, C_2) is entangleable. As far as we know, the problem stood open for the past 40 years.

Although our contribution concerns the foundations of quantum physics more than operational tasks or quantum computation, we believe it may be of strong interest to the community of QIP. It revisits a crucial notion in quantum information theory such as that of entanglement, promoting it to a universal feature of all non-classical theories that obey a few operational assumptions, and in doing so it solves an outstanding mathematical problem in the theory of ordered vector spaces.

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