Entangleability of cones

Guillaume Aubrun, Ludovico Lami*, Carlos Palazuelos, and Martin Plávala

arXiv:1911.09663 and arXiv:1910.04745

*Institut für Theoretische Physik und IQST, Universität Ulm, Albert-Einstein-Allee 11, D-89069 Ulm, Germany

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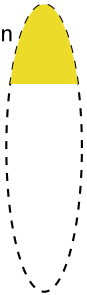
Philosophical introduction

- QM, lecture # 1: **superposition** is a property of the physical world.



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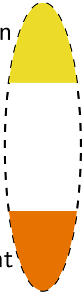
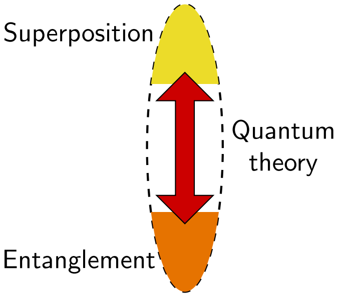


Philosophical introduction

- QM, lecture # 1: **superposition** is a property of the physical world.
- QM, lecture # 2: by superposing product states one obtains **entanglement**.
- These two fundamental notions seem to be related only by an 'accident' of the mathematical formalism.

  $\Longrightarrow$ We are allowed to deduce the existence of global entanglement from that of local superpositions *only* if we believe in the mathematical structure of quantum theory.



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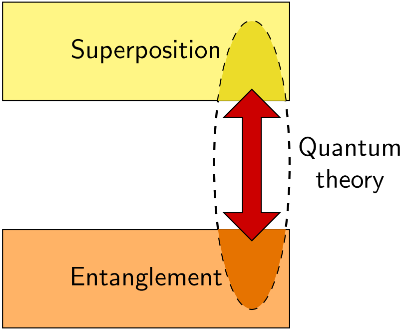

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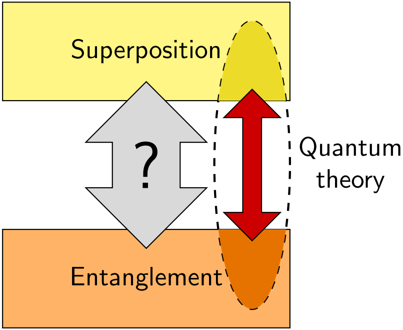
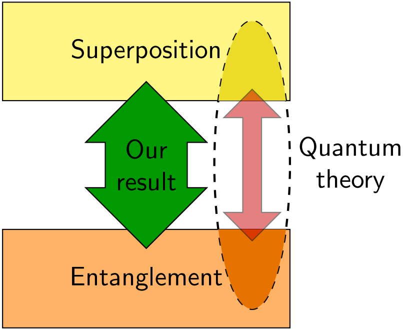


- However, quantum theory may not be the ultimate theory of nature.
- What happens to the relation between superposition and entanglement for other, possibly post-quantum theories?
- We fill this gap, and prove that (modulo some non-trivial assumptions)

superposition and entanglement logically imply one another.



Beyond quantum: a quick intro to GPTs

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Beyond quantum: a quick intro to GPTs

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- Two fundamental notions:
  1. **State:** a physical system together with a preparation procedure.
  2. **Effect:** a measurement device together with one of its possible outcomes.

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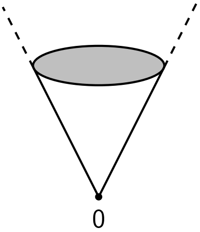
- For a systematic introduction see e.g. [LL, arXiv:1803.02902] or [Müller, arXiv:2011.01286].

Quantum theory of an n-level system as a GPT

- Un-normalised states: $n \times n$ positive semi-definite matrices PSD_n.

States

$n \times n$
Hermitian
matrices



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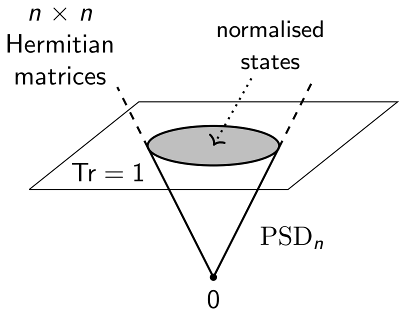

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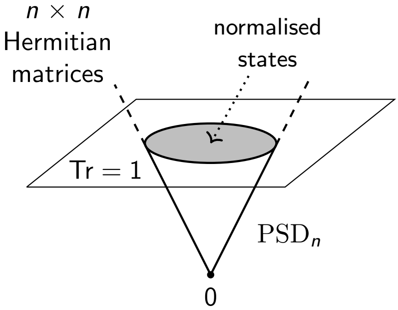

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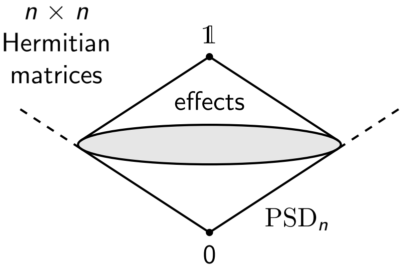

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- **Born rule**: $\Pr(E|\rho) = \mathrm{Tr}[\rho E] \in [0, 1]$.

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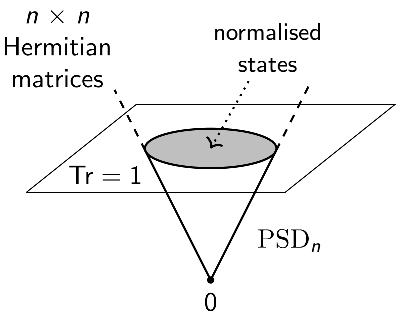

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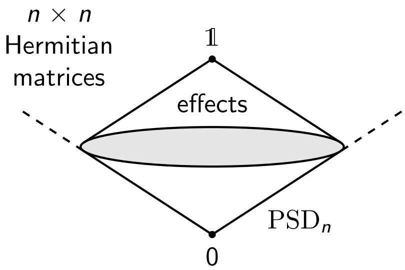
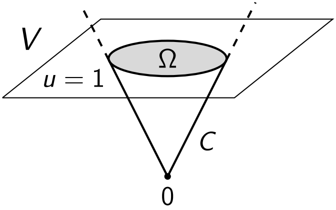


General probabilistic theories

- Un-normalised states: any (proper) convex cone $C \subseteq V$, where V is any finite-dim real vector space.



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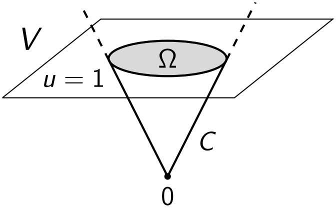

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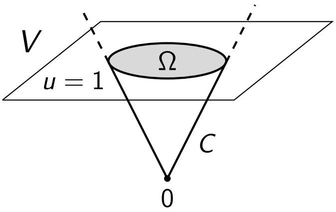

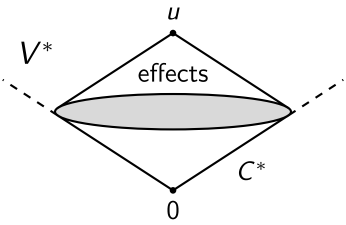

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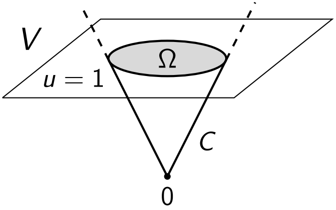

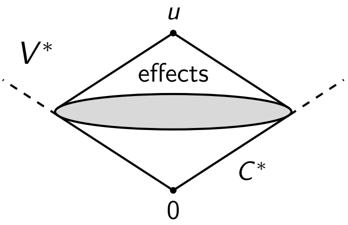

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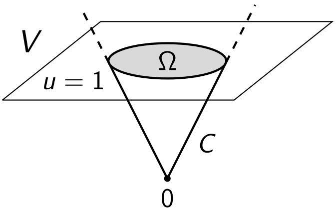
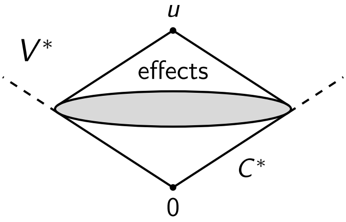
- Assumption # 1 (**no-restriction hypothesis**): *all mathematically reasonable effects are physically implementable.*





Example: classical theories

Real vector space V with *basis* $\{v_1, \ldots, v_d\}$. Then

$$C_{\mathrm{cl}} = \mathrm{cone}\{v_1, \ldots, v_d\} = \left\{ \sum\nolimits_i \lambda_i v_i :\, \lambda_i \geq 0\ \forall\, i \right\}$$

and any $u \in V^*$ such that $u(v_i) > 0\ \forall\, i$ define a **classical GPT** (V, C_{cl}, u)
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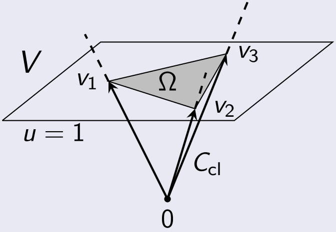

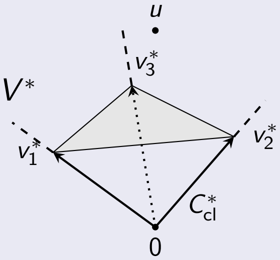

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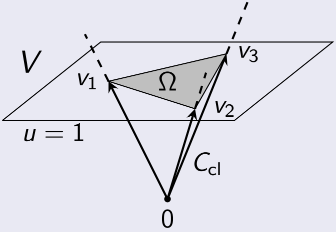

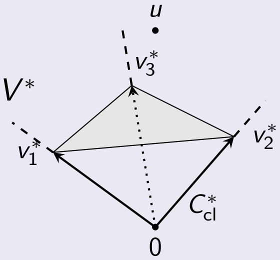

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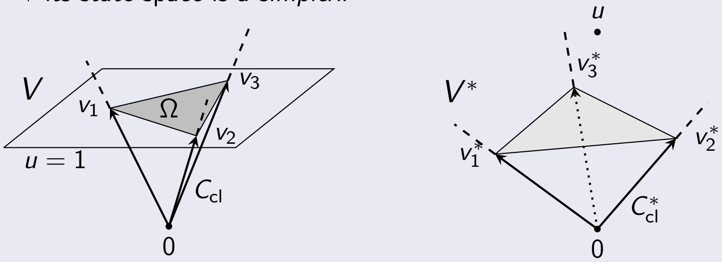

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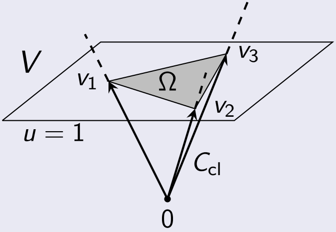

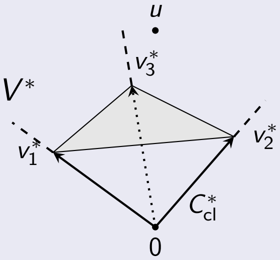

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- A cone that is not classical, in some sense, describes a GPT with some notion of superposition.

Bipartite systems in GPTs

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  2. Separable effects should yield probabilities when measured on C_{AB}:

$$C_{AB}^* \supseteq C_A^* \odot C_B^* =: (C_A \circledast C_B)^*\,.$$

This is about all we can say. We are left with the two relations

$$C_A \odot C_B \subseteq C_{AB}\,, \tag{1}$$

$$C_A^* \odot C_B^* = (C_A \circledast C_B)^* \subseteq C_{AB}^*\,. \tag{2}$$

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Note

If $C_A \odot C_B \neq C_A \circledast C_B$, then

- ▷ either $C_A \odot C_B \subsetneq C_{AB}$, that is, there are entangled states; or
- ▷ or $C_A^* \odot C_B^* \subsetneq C_{AB}^*$, that is, there are entangled measurements.

Main problem

Problem

When are C_A, C_B **entangleable**, i.e. satisfy $C_A \odot C_B \neq C_A \circledast C_B$?

[1] I. Namioka and R. R. Phelps, Pacific J. Math. **31**, 469–480 (1969)

[2] G. P. Barker, Linear Multilinear Algebra **4**, 191–199 (1976). G. P. Barker, Linear Algebra Appl. **39**, 263–291 (1981).

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  (Classical GPT + any other GPT $\longrightarrow$ no entanglement.)

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- Less trivial:[1]

  $$C_A \textit{ is classical} \iff C_A, C_B \textit{ are not entangleable for all cones } C_B.$$

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  C_A is classical $\Longleftrightarrow$ C_A, C_B are not entangleable for all cones C_B.
- *Barker's conjecture:*[2]

  C_A, C_B are entangleable $\Longleftrightarrow$ neither C_A nor C_B is classical.

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Main result

Theorem A (Solution of Barker's conjecture)

C_A, C_B are entangleable $\iff$ neither C_A nor C_B is classical.

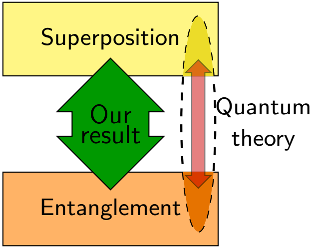
[3] T. Fritz et al., SIAM J. Appl. Algebra Geom. **1**, 556–574 (2017). B. Passer et al., J. Funct. Anal. **274**, 3197–3253 (2018).

Main result

Theorem A (Solution of Barker's conjecture)

C_A, C_B are entangleable $\Longleftrightarrow$ neither C_A nor C_B is classical.

- Classicality (or the absence of superposition) is a *local* property. Entangleability is a *global* property.



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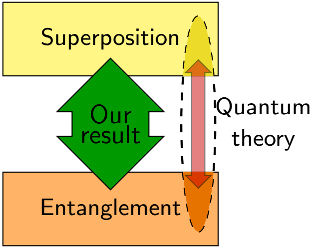

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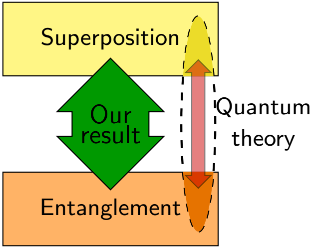

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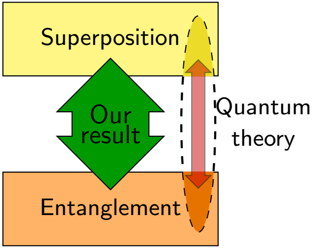

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- A conjecture in the theory of abstract operator systems[3] asked whether $C \odot \mathrm{PSD}_n = C \circledast \mathrm{PSD}_n$, where PSD_n is the cone of $n \times n$ positive semi-definite matrices ($n \geq 2$), happened only when C is classical $\longrightarrow$ **YES!**

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Overview of the proof

We already saw that C_A, C_B entangleable $\Longrightarrow$ neither of them is classical.
We prove the opposite implication.

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1. **Kite-square sandwiching**: we reduce the problem to a (modified) entangleability problem for two special 3-dimensional cones.
2. **Brute force construction** of an entangled state for this 3-dimensional problem.

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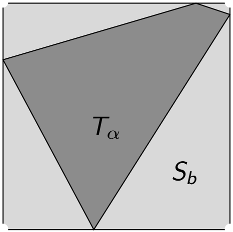
$$T_\alpha := \operatorname{conv}\{(1,\alpha_1),(\alpha_2,1),(-1,\alpha_3),(\alpha_4,-1)\}.$$

- The "blunt square"

$$S_b := [-1,1]^2 \setminus \{-1,1\}^2 \,.$$



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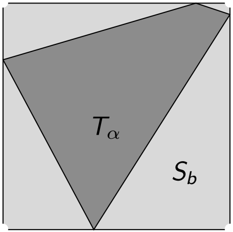

- Construct the 3-dimensional cones $\mathscr{C}(T_\alpha)$ and $\mathscr{C}(S_b)$, where

$$\mathscr{C}(K) := \mathrm{cone}\,\{x \oplus 1 :\ x \in K\}\,.$$

Definition (kite-square sandwiching)

A cone $C \subset V$ admits a kite-square sandwiching if and only if there are linear maps $\Phi_\ell : \mathbb{R}^3 \to V$ (lift) and $\Phi_c : V \to \mathbb{R}^3$ (compress) such that:

1. $\Phi_c \circ \Phi_\ell = \mathrm{id}_{\mathbb{R}^3}$;
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We are going to take this for granted. Instead, let us focus on the next question:

How does this help?

Suppose we prove the following crucial fact (intuition later):

Proposition (kites entangle strongly)

Fix $\alpha, \beta \in (-1,1)^4$. Then there is $\omega \in \mathscr{C}(T_\alpha) \circledast \mathscr{C}(T_\beta)$ such that $\omega \notin \mathscr{C}(S_b) \odot \mathscr{C}(S_b)$.

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- Let C_A, C_B be non-classical cones. Consider the lift-and-compress maps $\Phi_\ell^A, \Phi_\ell^B, \Phi_c^A, \Phi_c^B$ from Theorem B. Then define

$$\Omega := \left(\Phi_\ell^A \otimes \Phi_\ell^B\right)(\omega) \in \left(\Phi_\ell^A \otimes \Phi_\ell^B\right)\left(\mathscr{C}(T_\alpha) \circledast \mathscr{C}(T_\beta)\right) \subseteq C_A \circledast C_B \,.$$

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- If $C_A \circledast C_B = C_A \odot C_B$, and hence $\Omega \in C_A \odot C_B$, we arrive at a contradiction:

$$\omega = \Big(\underbrace{\Phi_c^A \circ \Phi_\ell^A}_{\mathrm{id}_{\mathbb{R}^3}} \otimes \underbrace{\Phi_c^B \circ \Phi_\ell^B}_{\mathrm{id}_{\mathbb{R}^3}}\Big)(\omega) = \left(\Phi_c^A \otimes \Phi_c^B\right)(\Omega) \in \mathscr{C}(S_b) \odot \mathscr{C}(S_b) \,.$$

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- How to prove the above Proposition? With a brute force approach. Four extreme point for each kite:

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- Rescale them to S_i, T_j so that

$$S_1 + S_3 = S_2 + S_4\,, \qquad T_1 + T_3 = T_2 + T_4\,.$$

- Construct ω:

$$\omega := S_1 \otimes T_2 - S_2 \otimes T_2 + S_2 \otimes T_1 + S_3 \otimes T_3 \overset{\text{easy}}{\in} \mathscr{C}(T_\alpha) \circledast \mathscr{C}(T_\beta)\,.$$

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- However, a daunting 37-line computation shows that we can make

$$F(\omega) \leq 0 \qquad \Longrightarrow \qquad \omega \notin \mathscr{C}(S_b) \odot \mathscr{C}(S_b)\,.$$

□

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