Mixing indistinguishable systems leads to a quantum Gibbs paradox

Benjamin Yadin, Benjamin Morris, Gerardo Adesso

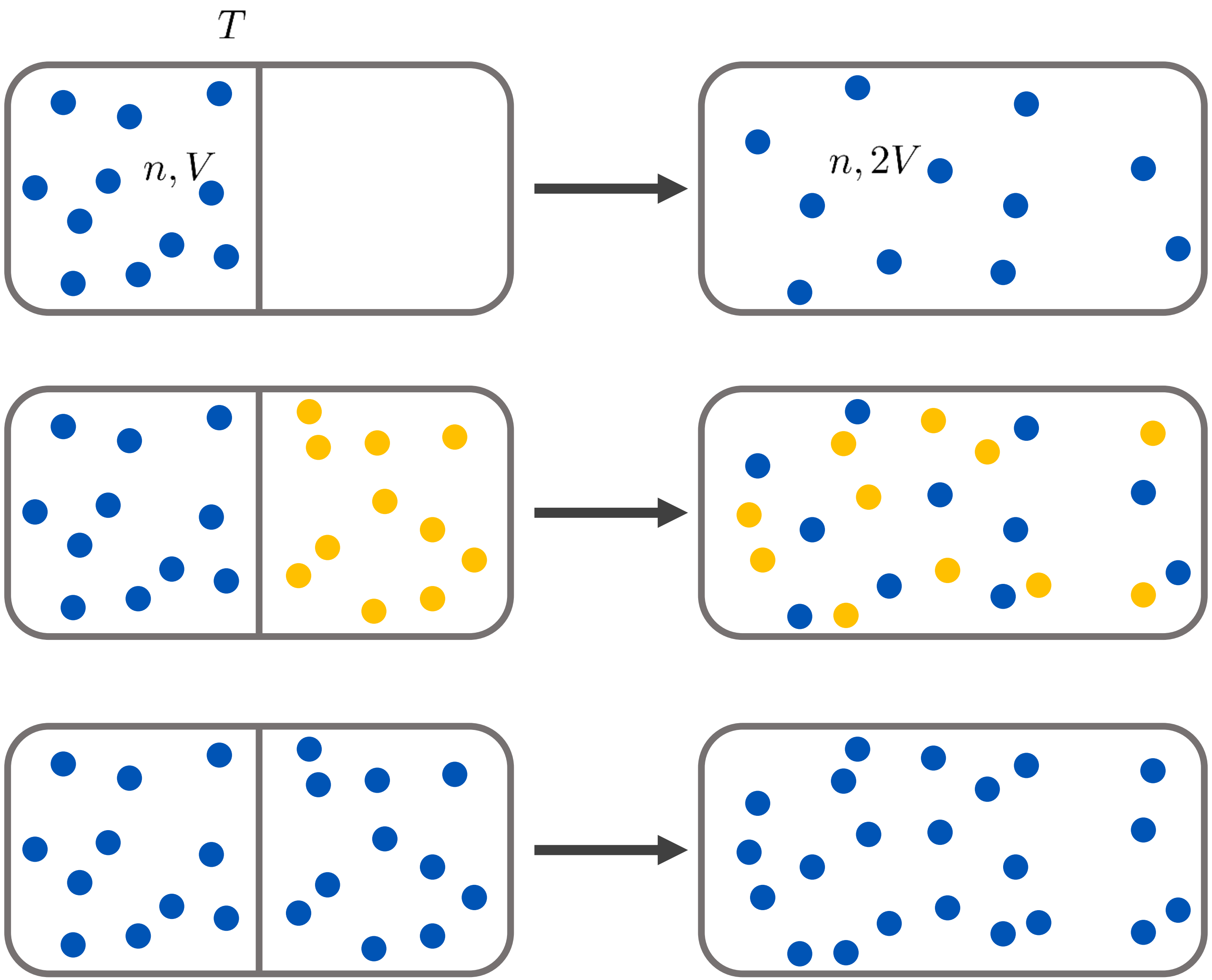
arXiv:2006.12482

University of Nottingham
UK | CHINA | MALAYSIA

UNIVERSITY OF OXFORD

The classical Gibbs paradox

Ideal gas in a box, expanding isothermally to twice its original volume



$\Delta S = n \ln 2$

$\Delta S = 2n \ln 2$

contradiction?

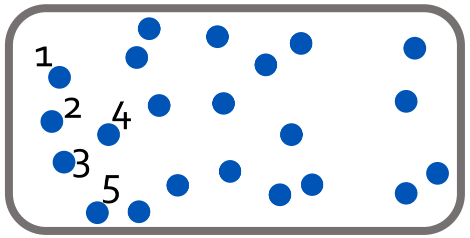
$\Delta S = 0$

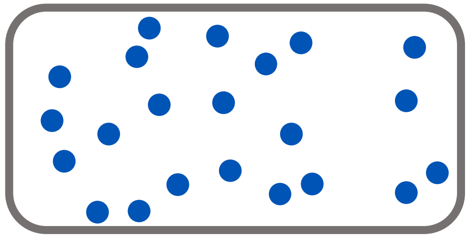
Entropy change depends on whether the gases are identical

The classical Gibbs paradox

Historically, seen as a problem with making entropy extensive

Gibbs + Boltzmann introduced a correction factor into microstate counting





$$\Omega = V^N,$$
$$S = \ln \Omega = N \ln V$$

$$\Omega = \frac{V^N}{N!},$$
$$S = \ln \Omega \approx N \ln \left(\frac{V}{N} \right) + N$$

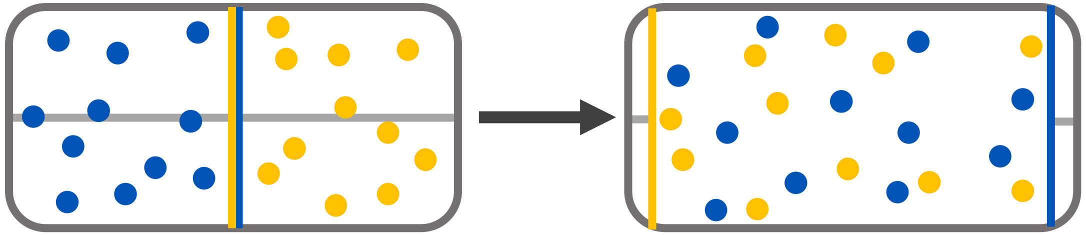
Removing particle labels means phase space volume is divided by $N!$

Simon Saunders, "The Gibbs Paradox," Entropy 20, 552 (2018)
Dennis Dieks, "The Gibbs Paradox and Particle Individuality," Entropy 20, 466 (2018)

What is an observer?

Informed observer

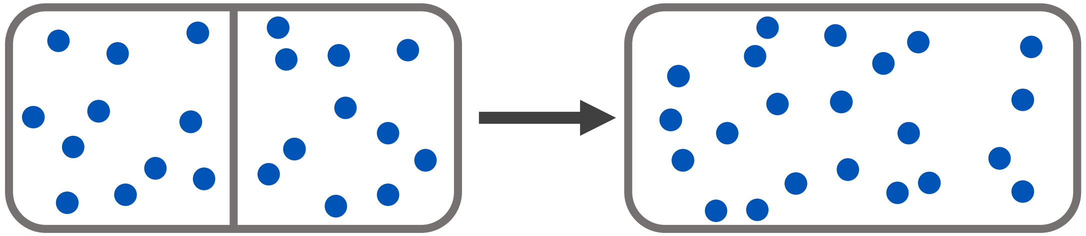
$\Delta S = 2n \ln 2$



Can extract work from each gas independently (semi-permeable membrane)

Ignorant observer

$\Delta S = 0$



Can't extract work – apparatus couples identically to both gases

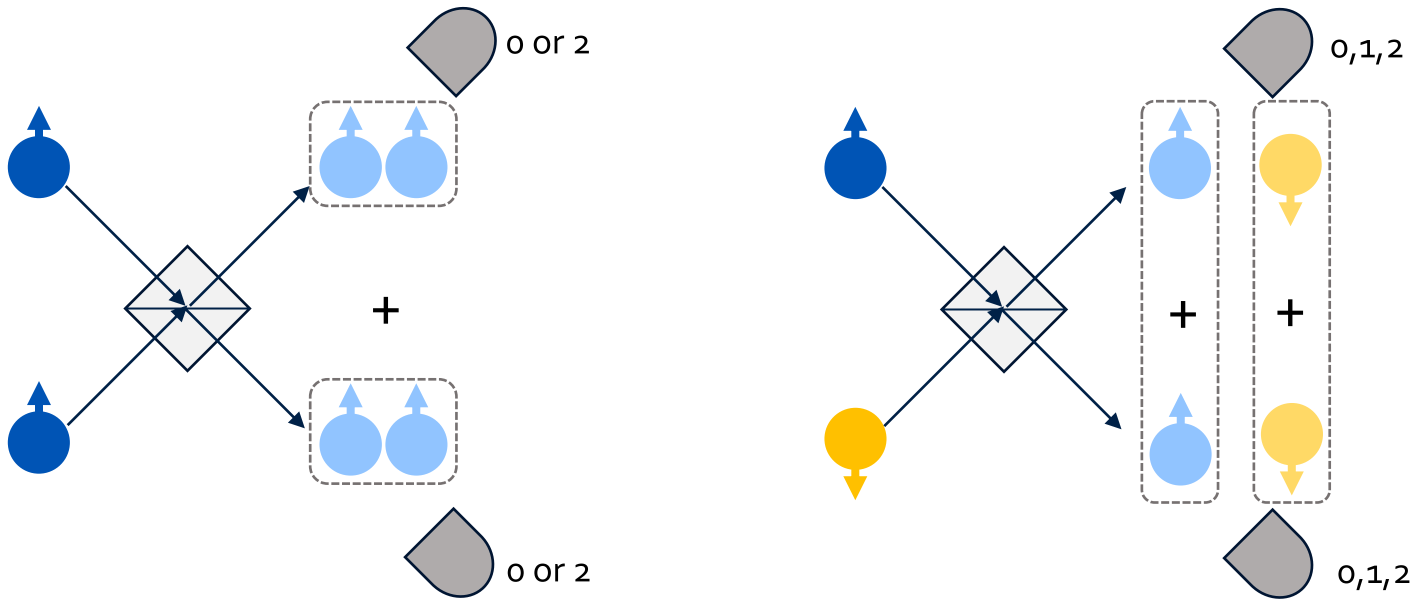
"observer" = designation of which degrees of freedom can be operated upon

Edwin T Jaynes, "The Gibbs paradox," in Maximum entropy and Bayesian methods (Springer, 1992)

> The amount of useful work that we can extract from any system depends – obviously and necessarily – on how much "subjective" information we have about its microstate, because that tells us which interactions will extract energy and which will not; this is not a paradox, but a platitude. If the entropy we ascribe to a macrostate did not represent some kind of human information about the underlying microstates, it could not perform its thermodynamic function of determining the amount of work that can be extracted reproducibly from that macrostate.

Why might quantum be different?

Recall the **Hong-Ou-Mandel effect** in quantum optics:



Non-polarising beam-splitter and number counting are able to tell if the polarisations are equal or opposite

These operations are polarisation-independent, so **accessible to the ignorant observer**

Recent / related works on thermodynamics with identical particles:

- Holmes et al., PRL 124, 210601 (2020); also NPJ 22, 113015 (2020)
- Watanabe et al., PRL 124, 210604 (2020)
- Myers and Deffner, PRE 101, 012110 (2020)
- Allahverdyan and Nieuwenhuizen, PRE 73, 066119 (2006)

Overview

We analyse the thought experiment in a **fully quantum** manner using bosons / fermions

Want to find the **fundamental limits** on work extractable by different observers

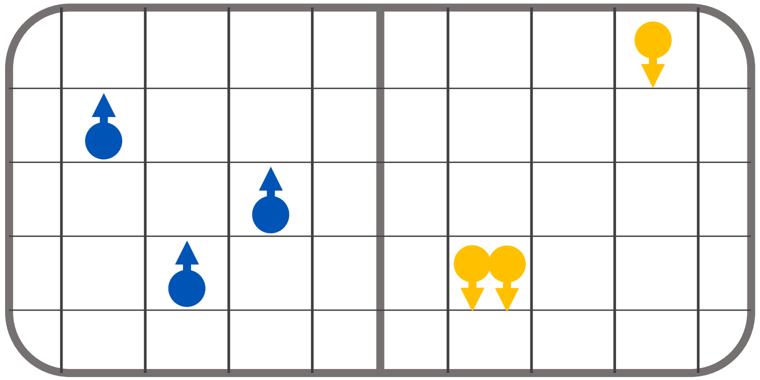
- Toy model
- Classical analysis: state-counting
- Hilbert space structure and the observers' operations
- Entropy changes and interesting limits

The model

Our model

A very simple model:

- 2 sides of a box, each with $\frac{d}{2}$ "cells"
- Start with n particles on each side
- Distinguish the gases by a "spin" – just a degree of freedom either ↑ or ↓
- $H = 0$ (all cells are degenerate in energy)
- Both sides initially thermalised



Difference between the observers:

- **Informed observer can interact with the spin**
- **Ignorant observer cannot – dynamics must be spin-independent**

Our model

Is the model too naïve?

- **How can you extract work if $H = 0$?**

  Couple the system to a heat bath B at temperature T and a work battery W.
  Total energy is conserved

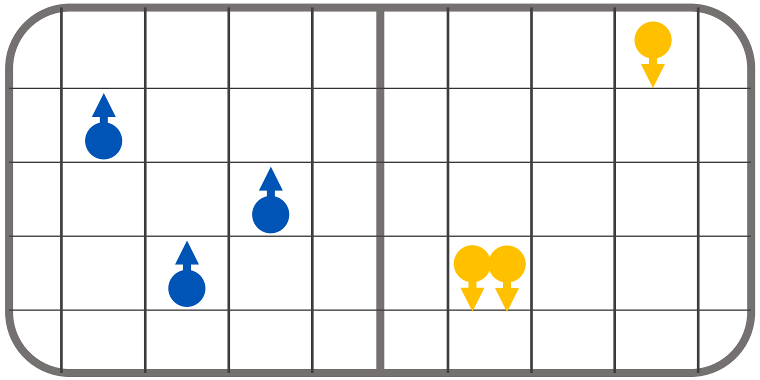
  Extracted work is proportional to entropy change:

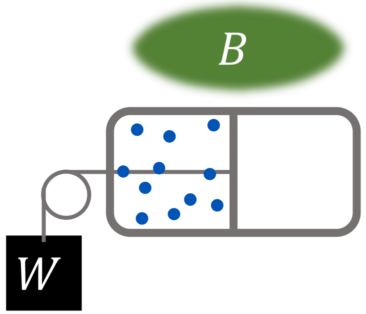
$$\Delta W = k_B T\,[S(\rho') - S(\rho)]$$

$$S(\rho) = -\operatorname{Tr}(\rho \ln \rho)$$

- **Does this really model an ideal gas?**

  Yes – recovers the correct classical entropy changes





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Classical case

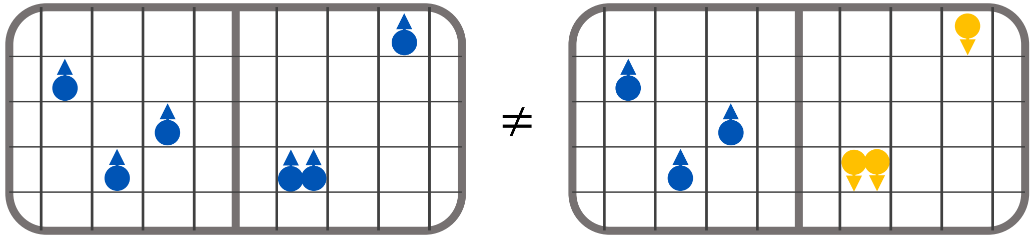
Classical case

Initial state: on each side, we have a uniform distribution of n identical particles over $d/2$ cells

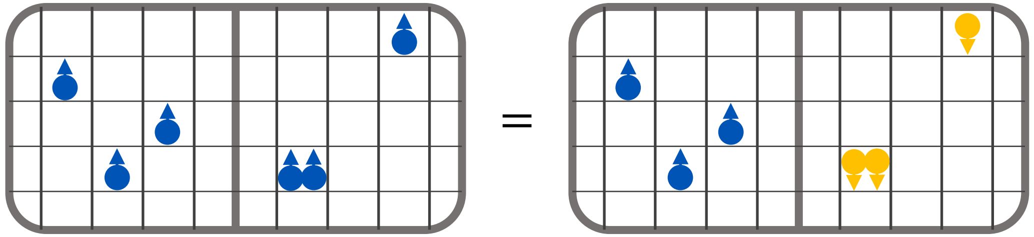
Final state: uniform distribution over **all** configurations of $2n$ particles over d cells

Entropy calculation boils down to a simple counting of microstates

Informed observer: counting depends on whether the spins are the same or different



Ignorant observer: different spin configurations are counted as the same



Classical case

ΔS from microstate counting

	Informed	Ignorant
Identical gases	$\ln\binom{2n+d-1}{2n} - 2\ln\binom{n+d/2-1}{n}$	$\ln\binom{2n+d-1}{2n} - 2\ln\binom{n+d/2-1}{n}$
Different gases	$2\ln\binom{n+d-1}{n} - 2\ln\binom{n+d/2-1}{n}$	$\ln\binom{2n+d-1}{2n} - 2\ln\binom{n+d/2-1}{n}$

Classical case – macroscopic limit

ΔS from microstate counting

Take the limit $n \gg 1$, then $d \to \infty$

Large particle number and low density

	Informed	Ignorant
Identical gases	≈ 0	≈ 0
Different gases	$2n \ln 2$	≈ 0

≈ 0 means $O(\ln n)$

Only this changes in quantum case (pointing to the Different gases / Ignorant cell)

Recovers the ideal gas results

The quantum ignorant observer

Quantum case: Hilbert space structure

Our task is to describe the effective state space seen by the ignorant observer

Single-particle Hilbert space:

$$\mathcal{H}_1 = \mathcal{H}_x \otimes \mathcal{H}_s$$

"position" (cell) $|i\rangle\,,\ i = 1, \ldots, d$

"spin" $|\uparrow\rangle\,, |\downarrow\rangle$

Single-particle basis change $u^{\otimes N}$
reps. of $\mathrm{U}(d),\ \mathrm{U}(2)$

N-particle Hilbert space for bosons / fermions:

$$\mathcal{H}_N = P_\pm \left(\mathcal{H}_x^{\otimes N} \otimes \mathcal{H}_s^{\otimes N}\right)$$

$P_\pm$ is projector onto (anti-)symmetric subspace

Permutations of particle labels act simultaneously on space and spin:

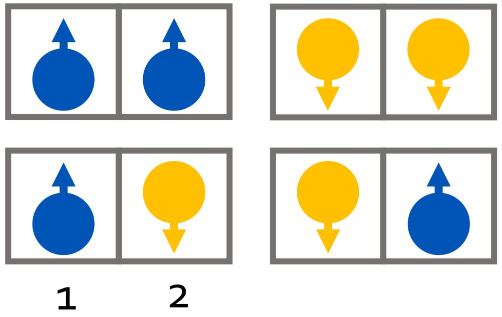
$$\Pi = \Pi_x \otimes \Pi_s$$

rep. of S_N

Coupling spin and spatial symmetry

Spin and spatial permutation symmetries must combine to give overall (anti-)symmetry

Familiar example from atomic physics:
(2 particles, each in its own cell)



$$\frac{|1\,2\rangle + |2\,1\rangle}{\sqrt{2}} \quad \xleftrightarrow{\text{bosons}} \quad |\uparrow\uparrow\rangle,\ |\downarrow\downarrow\rangle,\ \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad J = 1$$

fermions

$$\frac{|1\,2\rangle - |2\,1\rangle}{\sqrt{2}} \quad \longleftrightarrow \quad \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad J = 0$$

A particular symmetry of the spatial wavefunction comes with each *J*

$J(J+1)$ = eigenvalue of $\left(\sum_{i=1}^{N} J_x^{(i)}\right)^2 + \left(\sum_{i=1}^{N} J_y^{(i)}\right)^2 + \left(\sum_{i=1}^{N} J_z^{(i)}\right)^2$

State space of ignorant observer

Spatial and spin representations are linked via *J*

In general: $\mathcal{H}_N = \bigoplus_{J=0}^{N/2} \mathcal{H}_x^J \otimes \mathcal{H}_s^J$

$\mathcal{H}_x^J$: $\mathrm{U}(d)$ irrep; $\mathcal{H}_s^J$: $\mathrm{U}(2)$ irrep

Ignorant observer acts on this part only; *J* is conserved

(Schur-Weyl duality for groups $\mathrm{U}(d)$ and S_N, used twice)

Adamson et al., PRA 78, 033832 (2008)

Conditions on the global unitary U (coupling system, heat bath, work battery):

- U acts only on $\mathcal{H}_x^J$ factors
- Must preserve exchange symmetry: $[U, \Pi] = 0 \ \forall \Pi$

Tracing out the spin part, the ignorant observer works with the state $\rho_x = \mathrm{Tr}_s \, \rho = \bigoplus_J p_J \rho_x^J$

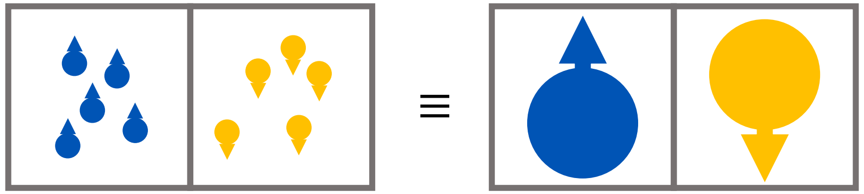
Each component ρ_x^J in the ensemble evolves independently in the space $\mathcal{H}_x^J$ of dimension d_J

Entropy changes

Thermalisation by the quantum ignorant observer:

$$\rho_x = \mathrm{Tr}_s\, \rho = \bigoplus_J p_J \rho_x^J \to \bigoplus_J p_J \frac{I_x^J}{d_J}$$

not maximally mixed! (due to conservation law)



p_J from Clebsch-Gordan coefficients (two large spins);
d_J from rep. theory formulas

$$\Delta S_{\mathrm{igno}} = \sum_J p_J \ln d_J - 2 \ln \binom{n + d/2 - 1}{n}$$

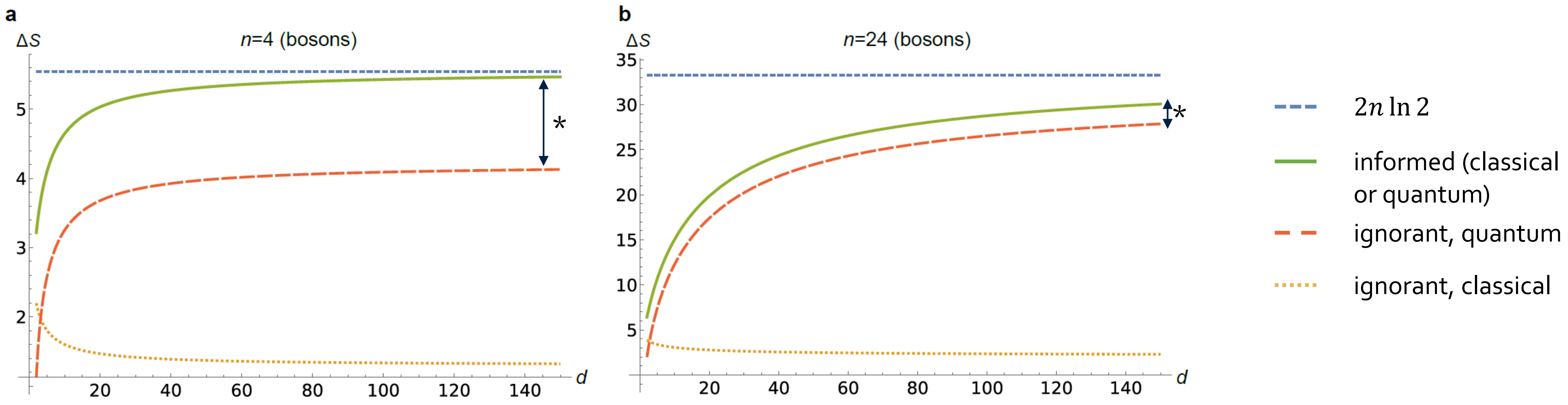
Never larger than the informed observer: $\Delta S_{\mathrm{igno}} \leq \Delta S_{\mathrm{info}} = 2 \ln \binom{n + d - 1}{n} - 2 \ln \binom{n + d/2 - 1}{n}$

But can be above the classical ignorant observer!

Translates into an average extracted work $\sum_J p_J W_J = k_B T \Delta S$ (achievable with thermal operations)

Horodecki & Oppenheim., Nat. Comm. 4, 2059 (2013)

Interesting limits



Low density limit: $d \to \infty$

$$\Delta S_{\text{info}} - \Delta S_{\text{igno}} \approx H(\mathbf{p}) - \frac{n^2}{2d^2} \quad *$$

$$H(\mathbf{p}) = -\sum_J p_J \ln p_J \approx \frac{1}{2} \ln n$$ for large n

Also taking **large n**:

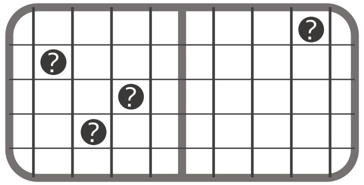
$$\Delta S_{\text{info}} \approx \Delta S_{\text{igno}} \approx 2n \ln 2$$

macro limit ≠ classical!

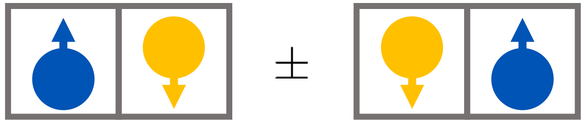
(and negligible fluctuations)

Why does it work?

Classically, for a given cell configuration, different spin configurations are indistinguishable by the ignorant observer

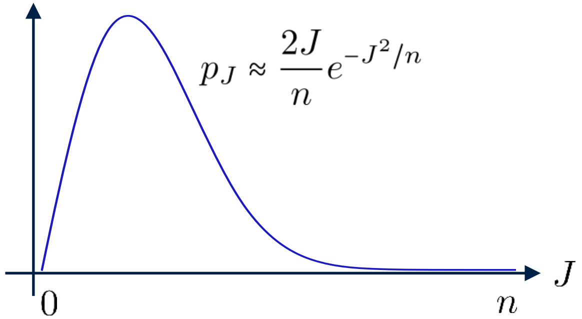


But certain **superpositions of spin configurations are distinguishable:**



Schur-Weyl basis states $|J, q\rangle_x$ for $\bigoplus_J \mathcal{H}_x^J$

In the macro limit with orthogonal spins, $\ln \dim \mathcal{H}_x^J \approx N \ln d$ with prob. $\to 1$



Rotation between "classical" basis and the Schur-Weyl basis can be accomplished to error ϵ by a circuit polynomial in $N, d, \ln(\epsilon^{-1})$

Bacon et al., PRL 97, 170502 (2006)

Variant of the quantum Fourier transform

Conclusions

- With quantum particles, relational spin information imprinted upon observable degrees of freedom
- Superpositions of classically indistinguishable configurations are distinguishable
- In a macroscopic limit, the lack of knowledge is no barrier – as much work can be extracted as if the particles were fully distinguishable
- Allowing fully quantum control, classical thermodynamics does not emerge in the macroscopic limit

Open questions:

- More realistic models, $H \neq 0$
- Understanding / approximating optimal operations
- Experimental proposals

Thank you

arXiv:2006.12482

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Thermalisation by ignorant observer - example

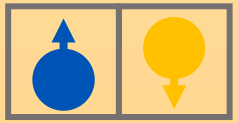
Ignorant observer wants to maximise entropy change for $\bigoplus_J p_J \rho_x^J \to \bigoplus_J p_J {\rho'_x}^J$ $\qquad \Delta S_{\text{igno}} = \sum_J p_J \Delta S^J$

Therefore wants to make each ${\rho'_x}^J$ **maximally mixed**

Example: $n = 1, d = 2$

initially

$$|\psi\rangle = \frac{1}{\sqrt{2}} \cdot \underbrace{\frac{|1\,2\rangle + |2\,1\rangle}{\sqrt{2}} \cdot \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}}_{J=1} + \frac{1}{\sqrt{2}} \cdot \underbrace{\frac{|1\,2\rangle - |2\,1\rangle}{\sqrt{2}} \cdot \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}}_{J=0}$$

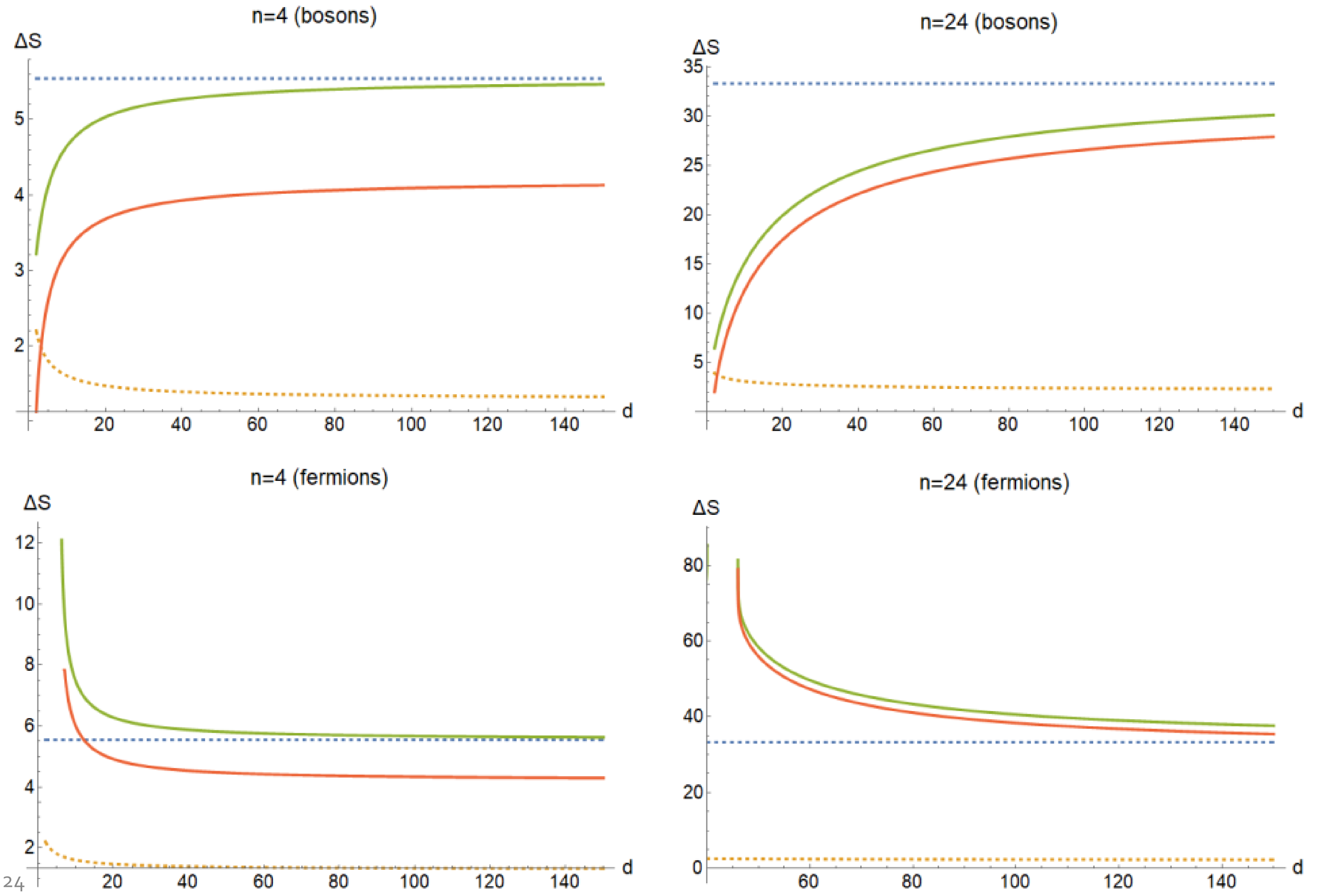


(unchanged)

mixture of $\frac{|1\,2\rangle + |2\,1\rangle}{\sqrt{2}}, \; |1\,1\rangle, \; |2\,2\rangle$

$$\Delta S_{\text{igno}} = \frac{1}{2} \ln 3 + \frac{1}{2} \ln 1$$

Bosons versus Fermions



$2n \ln 2$

informed (classical or quantum)

ignorant, quantum

ignorant, classical

For $d = 2n \gg 1$

$$\Delta S_{\text{info}} \approx \Delta S_{\text{igno}}$$
$$\approx 4n \ln 2$$

(twice classical)

Work fluctuations

Work isn't extracted deterministically

Each J occurs with probability p_J and results in entropy change $\Delta S_{\text{igno}}(J) = \ln d_J - 2\ln\binom{n+d/2-1}{n}$

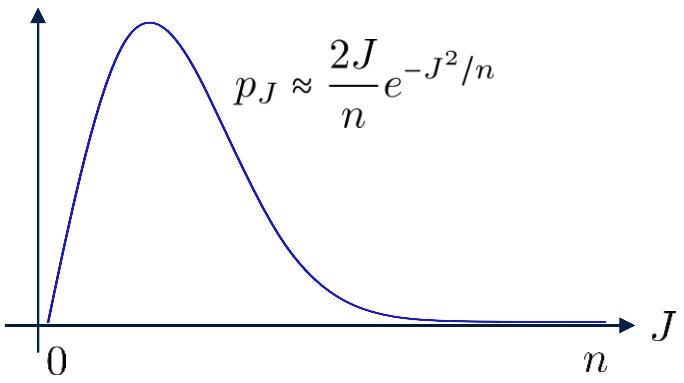
with average value $\Delta S_{\text{igno}} = \sum_J p_J \Delta S_{\text{igno}}(J)$

Look at variance: $V(\Delta S_{\text{igno}}) = \sum_J p_J \Delta S_{\text{igno}}(J)^2 - \Delta S_{\text{igno}}^2$

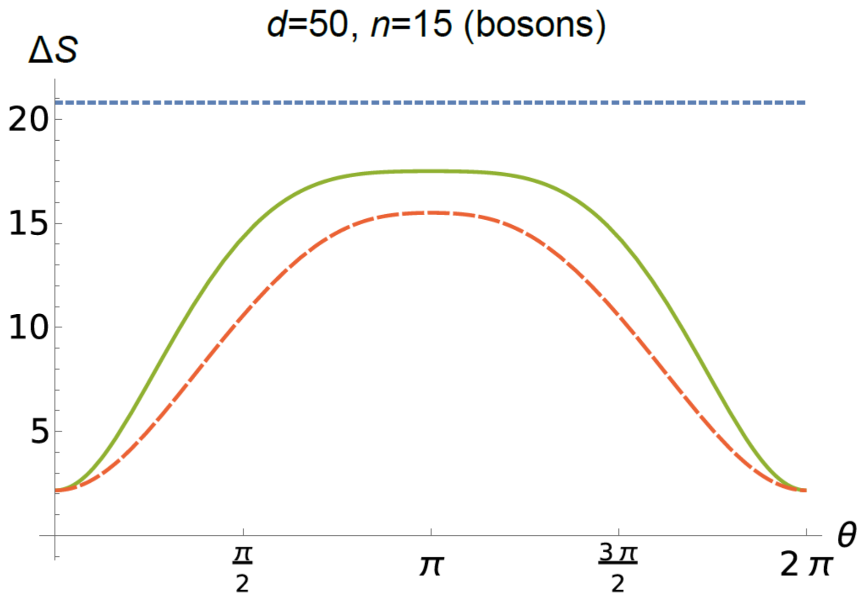
Large number, low density limit:

Mean is $\Delta S_{\text{igno}} \approx 2n\ln 2$

Variance is $V(\Delta S_{\text{igno}}) \approx \frac{\pi^2}{24} \approx 0.411$ **negligible compared with mean**



Partial distinguishability



$2n \ln 2$

informed (classical or quantum)

ignorant, quantum

Spin states: $|\uparrow\rangle$ on left, $\cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle$ on right

Informed observer can perform any $\bigoplus_M U_{xsBW}^{(M)}, \quad M = \frac{N_\uparrow - N_\downarrow}{2}$