No Quantum Speedup over Gradient Descent

Lower Bounds for Convex Optimization

Ankit Garg [1] Robin Kothari [2] Praneeth Netrapalli [1] Suhail Sherif [1,3]

[1] Microsoft Research India

[2] Microsoft Quantum and Microsoft Research

[3] Tata Institute of Fundamental Research

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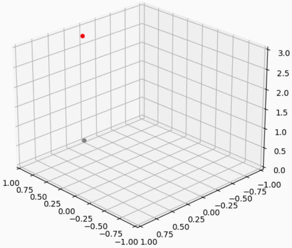

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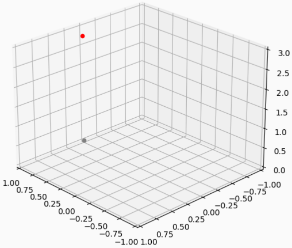

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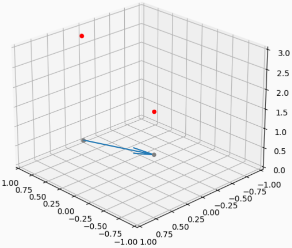

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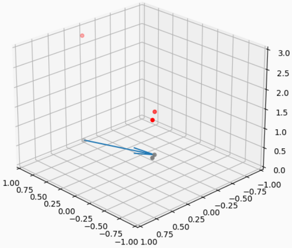

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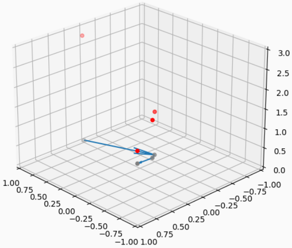

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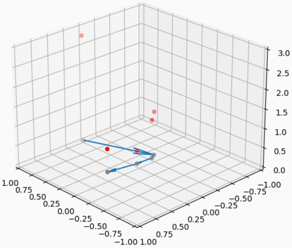

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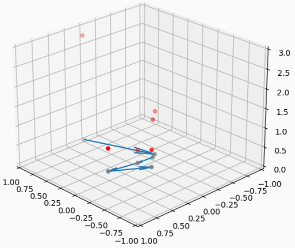

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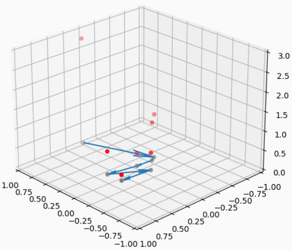

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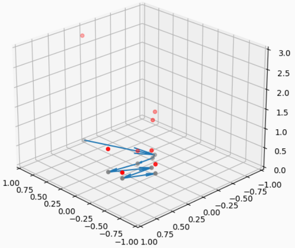

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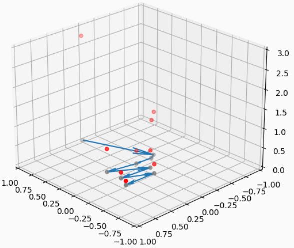

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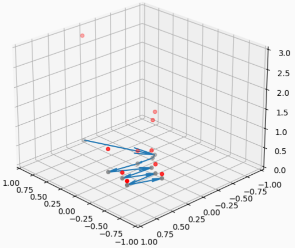

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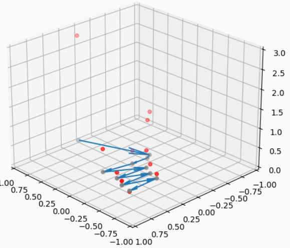

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- Abstracting out Gradient Descent:
  - Function value oracle
  - Function gradient oracle

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Use First-Order Convex Optimization as a proxy for Gradient Descent.

First-Order Convex Optimization

The Task

Given: a convex region B, first-order oracle access to a convex function $f : \mathbb{R}^n \to \mathbb{R}$.

On input x, oracle O_f returns $f(x), \nabla f(x)$.

The Task

Given: a convex region B, first-order oracle access to a convex function $f : \mathbb{R}^n \to \mathbb{R}$.

Find $x^* = \arg\min_{x \in B} f(x)$.

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Given: a convex region B, first-order oracle access to a convex function $f : \mathbb{R}^n \to \mathbb{R}$.

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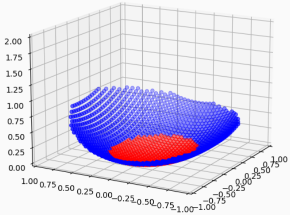

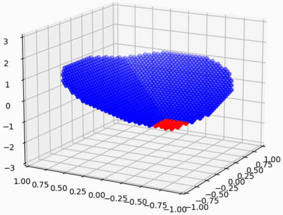

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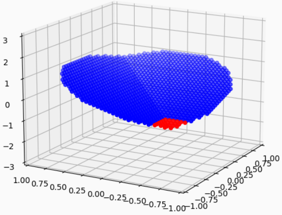

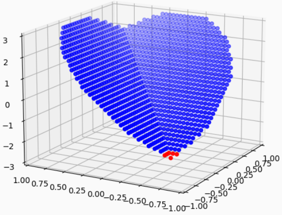

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ϵ-optimal for G-Lipschitz function in ball of radius R

$\iff$

ϵ/GR-optimal for 1-Lipschitz function in ball of radius 1

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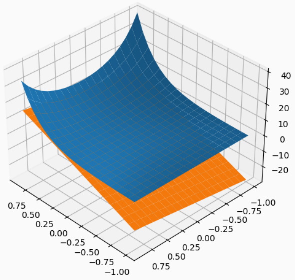

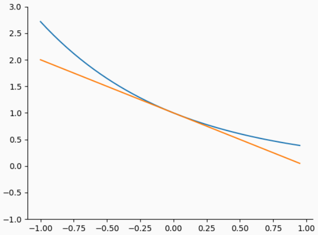

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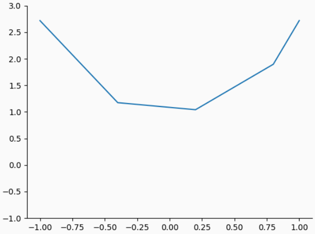

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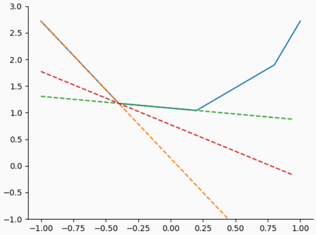

$g \in \nabla f(x) \Leftrightarrow f(x+v) \geq f(x) + \langle v, g \rangle$ for all v

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Center of Gravity Method

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$n \log(1/\epsilon)$ steps.

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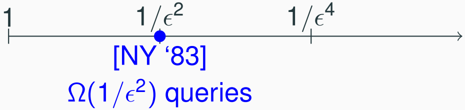

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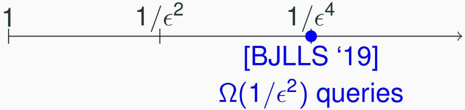

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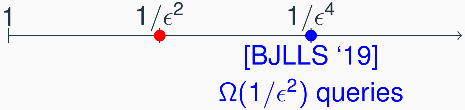

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Theorem (Garg-Kothari-Netrapalli-S. ‘20)

For any $\epsilon > 0$, there is a family of 1-Lipschitz functions $\{f : \mathbb{R}^n \rightarrow \mathbb{R}\}$ with $n = \Theta(1/\epsilon^2)$ such that any randomized algorithm solving first-order convex optimization on these requires $\Omega(1/\epsilon^2)$ queries.

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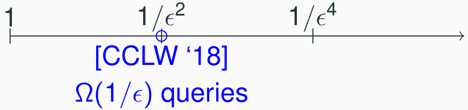

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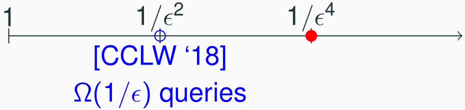

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- Each query should access the information in a controlled manner.

Lower Bounds

Randomized Lower Bound

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If x_i is a maximum, then e_i is a subgradient.

Function Class

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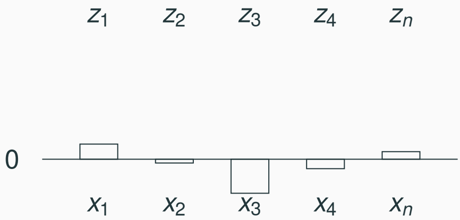

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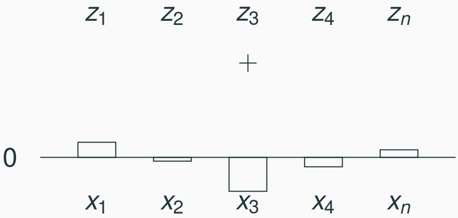

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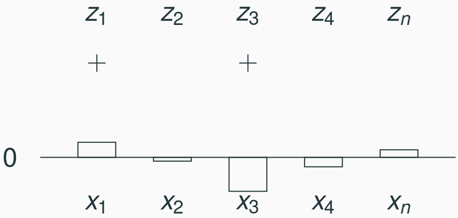

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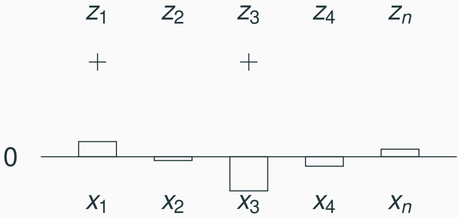

$\approx$ 2 bits of z revealed per query.

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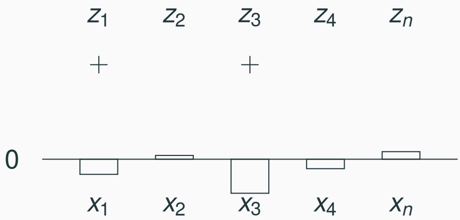

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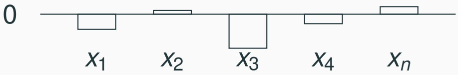
$+$	$-$	$+$	$+$	$-$



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$$z \in \{+1, -1\}^n$$

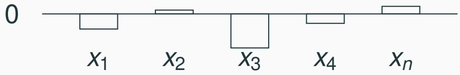
$$f_z(x) = \max\{z_1 x_1, z_2 x_2, \ldots, z_n x_n\}.$$

$$\text{Minimum} = -\frac{1}{\sqrt{n}}, \text{ at } x = \left(-\frac{z_1}{\sqrt{n}}, \ldots, -\frac{z_n}{\sqrt{n}}\right).$$

$$\text{Set } \epsilon = \frac{.9}{\sqrt{n}}.$$

The behaviour of f

z_1	z_2	z_3	z_4	z_n
$+$	$-$	$+$	$+$	$-$



Finding ϵ-optimal point $\implies$ learning z.

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Set $\epsilon = \frac{.9}{\sqrt{n}}$.

Requires $\Omega(n) = \Omega(1/\epsilon^2)$ queries.

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A quantum algorithm can ϵ-optimize the previous function class in $O(\sqrt{n})$ queries.

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For any $S \subseteq [n]$, can find x such that

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Can then use Belovs' algorithm to learn z from such OR queries.

Lower Bounds

Quantum Lower Bound

The Query Model

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- For f, f' s.t. $f(x) = f'(x)$ and $\nabla f(x) = \nabla f'(x)$:

$$O_f |x\rangle_{INPUT} |\phi\rangle_{REST} = O_{f'} |x\rangle_{INPUT} |\phi\rangle_{REST}$$

The Base Function

"Complexity of Highly Parallel Non-Smooth Convex Optimization"
- Sébastien Bubeck, Qijia Jiang, Yin Tat Lee, Yuanzhi Li, Aaron Sidford

The Base Function

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$f(x) = \max\{x_1, x_2 - \gamma, x_3 - 2\gamma, \ldots, x_k - (k-1)\gamma\}.$$

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γ is small.

Minimum $\approx -\frac{1}{\sqrt{k}}$, at $x \approx \left(-\frac{1}{\sqrt{k}}, \ldots, -\frac{1}{\sqrt{k}}, 0, 0, \cdots\right)$.

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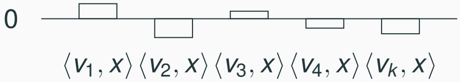

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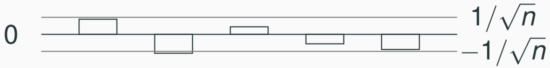

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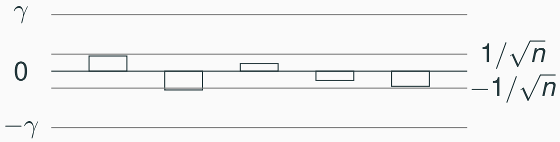

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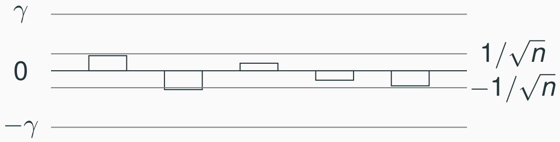

Whp, first query reveals v_1.

v_2 through v_k nearly random from $n-1$ dimensional space.

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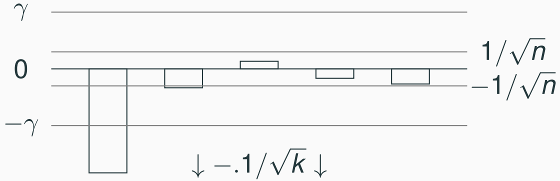

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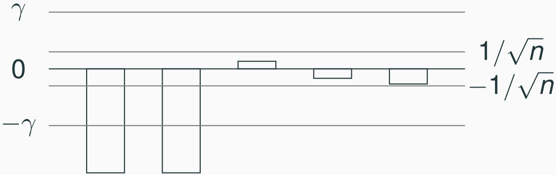

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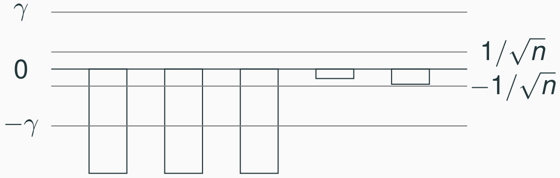

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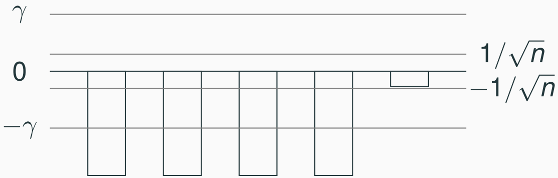

v_k still nearly at random from $n - k$ dimensional space.

Can't output ϵ-optimal point.

The Hybrid Argument

First
query

$$
\begin{aligned}
&|x_1\rangle|\phi_1\rangle \\
+&|x_2\rangle|\phi_2\rangle \\
+&|x_3\rangle|\phi_3\rangle \\
+&|x_4\rangle|\phi_4\rangle \\
+&|x_5\rangle|\phi_5\rangle \\
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$$+\cdots$$

$$\xrightarrow{\text{pass through oracle for } f_V(x)}$$

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First query

$$|x_1\rangle|\phi_1\rangle + |x_2\rangle|\phi_2\rangle + |x_3\rangle|\phi_3\rangle + |x_4\rangle|\phi_4\rangle + |x_5\rangle|\phi_5\rangle + \cdots$$

$$\xrightarrow{\text{pass through oracle for } f_V(x)}$$

First answer

$$|x_1\rangle|\psi_1\rangle + |x_2\rangle|\psi_2\rangle + |x_3\rangle|\psi_3\rangle + |x_4\rangle|\psi_4\rangle + |x_5\rangle|\psi_5\rangle + \cdots$$

The Hybrid Argument

First query

$$|x_1\rangle|\phi_1\rangle + |x_2\rangle|\phi_2\rangle + |x_3\rangle|\phi_3\rangle + |x_4\rangle|\phi_4\rangle + |x_5\rangle|\phi_5\rangle + \cdots$$

$$\xrightarrow{\text{pass through oracle for } f_{(v_1)}(x)=\langle v_1, x\rangle}$$

Corrupted answer

$$|x_1\rangle|\psi_1\rangle + |x_2\rangle|\psi_2\rangle + |x_3\rangle|\psi_3\rangle + |x_4\rangle|\psi_4'\rangle + |x_5\rangle|\psi_5\rangle + \cdots$$

The Hybrid Argument

First query		Corrupted answer
$\lvert x_1\rangle\lvert\phi_1\rangle$		$\lvert x_1\rangle\lvert\psi_1\rangle$
$+\lvert x_2\rangle\lvert\phi_2\rangle$		$+\lvert x_2\rangle\lvert\psi_2\rangle$
$+\lvert x_3\rangle\lvert\phi_3\rangle$	$\xrightarrow{\text{pass through oracle for } f_{(v_1)}(x)=\langle v_1,x\rangle}$	$+\lvert x_3\rangle\lvert\psi_3\rangle$
$+\lvert x_4\rangle\lvert\phi_4\rangle$		$+\lvert x_4\rangle\lvert\psi'_4\rangle$
$+\lvert x_5\rangle\lvert\phi_5\rangle$		$+\lvert x_5\rangle\lvert\psi_5\rangle$
$+\cdots$		$+\cdots$

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$+\|x_3\rangle\|\phi_3\rangle$	$\xrightarrow{\text{pass through oracle for } f_{(v_1)}(x)=\langle v_1, x\rangle}$	$+\|x_3\rangle\|\psi_3\rangle$
$+\|x_4\rangle\|\phi_4\rangle$		$+\|x_4\rangle\|\psi_4'\rangle$
$+\|x_5\rangle\|\phi_5\rangle$		$+\|x_5\rangle\|\psi_5\rangle$
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- Changing oracle #1 barely changes the resulting state after 1 query. (with high probability)
- The actual, corrupted states at the end are also close.

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- Changing oracle #1 barely changes the resulting state after 1 query. (with high probability)
- The actual, corrupted states at the end are also close.
- Actual, corrupted algorithm nearly the same.

The Hybrid Argument: Second query

Second query (corrupted)

$$
\begin{aligned}
&|x_1\rangle|\tau_1\rangle \\
+&|x_2\rangle|\tau_2\rangle \\
+&|x_3\rangle|\tau_3\rangle \\
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+&\cdots
\end{aligned}
\qquad \xrightarrow{\text{pass through oracle for } f_V(x)}
$$

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$$|x_1\rangle|\tau_1\rangle + |x_2\rangle|\tau_2\rangle + |x_3\rangle|\tau_3\rangle + |x_4\rangle|\tau_4\rangle + |x_5\rangle|\tau_5\rangle + \cdots$$

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Second answer (corrupted)

$$|x_1\rangle|\chi_1\rangle + |x_2\rangle|\chi_2\rangle + |x_3\rangle|\chi_3\rangle + |x_4\rangle|\chi_4\rangle + |x_5\rangle|\chi_5\rangle + \cdots$$

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$$\xrightarrow{\text{pass through oracle for } f_{(v_1, v_2)}(x) = \max\{\langle v_1, x\rangle, \langle v_2, x\rangle - \gamma\}}$$

Second answer (twice corrupted)

$$|x_1\rangle|\chi_1\rangle + |x_2\rangle|\chi_2'\rangle + |x_3\rangle|\chi_3\rangle + |x_4\rangle|\chi_4\rangle + |x_5\rangle|\chi_5\rangle + \cdots$$

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- Changing oracle #2 barely changes the resulting state after 2 queries. (with high probability)

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Modifications taken from Bubeck et al. can bring n down to $1/\epsilon^4$.

Accelerated Gradient Descent

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Similar proof to the one shown, but the function requires smoothing.

Open Problems

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Yet...

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- What is the quantum complexity of optimizing the function class

$$f_V(x) = \max\{\langle v_1, x\rangle, \langle v_2, x\rangle, \ldots, \langle v_k, x\rangle\}?$$