(Sub)Exponential advantage of adiabatic quantum computation with no sign problem

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Adiabatic quantum computation – background

- A model of quantum computation utilizing a slowly changing Hamiltonian $H(s)$: The adiabatic evolution stays in the ground state $\psi(s)$ and maps $\psi(0)$ to $\psi(1)$.

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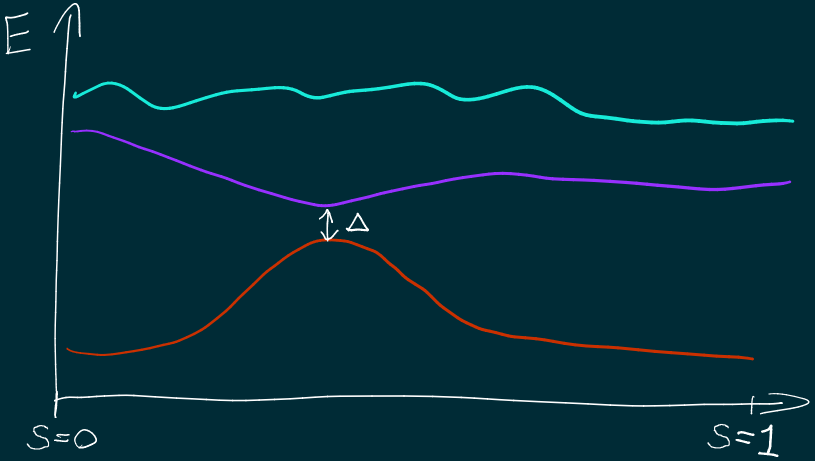

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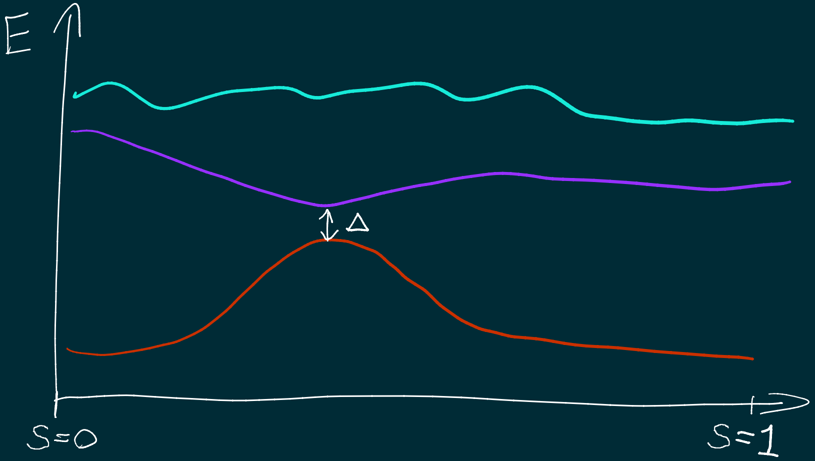

- The ground state is accurately mapped if the evolution time is $t = \Theta(1/\mathrm{poly}(\Delta))$

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- An interesting special case is when the Hamiltonians have “no sign problem”, i.e., all off-diagonal matrix elements of $H(s)$ are non-positive for every $s \in [0,1]$ (note that this is a basis-dependent property)

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- However, there are topological obstruction to path integral Monte Carlo [Hastings & Freedman, 2013]
- Diffusion Monte Carlo is sensitive to ℓ_1 vs. ℓ_2 differences in the ground state [Jarret, Jordan, & Lackey, 2016]

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- Long cycles that cannot be approximated by shorter cycles are problematic (especially if the long cycles make important contribution, e.g., particle on a torus)
- In practice this issue can be often fixed by adding appropriate non-local moves

Exponential quantum advantage with no sign problem

Main result (informal)

There is a family of sparse sign-problem-free Hamiltonians on n qubits with a straight adiabatic path featuring a spectral gap $\Delta = \Omega(1/\mathrm{poly}(n))$, whose evolution requires $2^{\sqrt[5]{n}}$ queries to the Hamiltonian entries to simulate classically.

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The corresponding graph problem (informal)

Given (via black-box access) a sparse graph on 2^n vertices and a specific "ENTRANCE" vertex, we can find the "EXIT" vertex in $\mathrm{poly}(n)$ time via adiabatic evolution of a corresponding Hamiltonian which has $1/\mathrm{poly}(n)$ spectral gap and no sign problem. At the same time any classical randomized algorithm must make at least $2^{\sqrt[5]{n}}$ queries to the graph (i.e., to the black-box) for finding the "EXIT".

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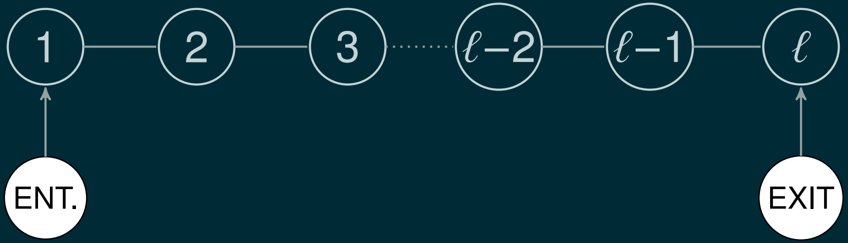

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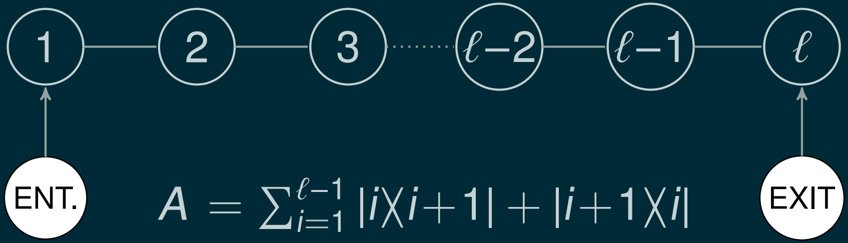

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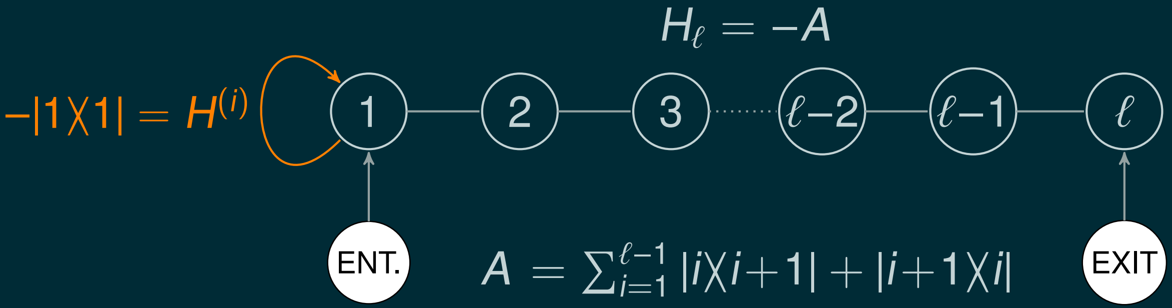

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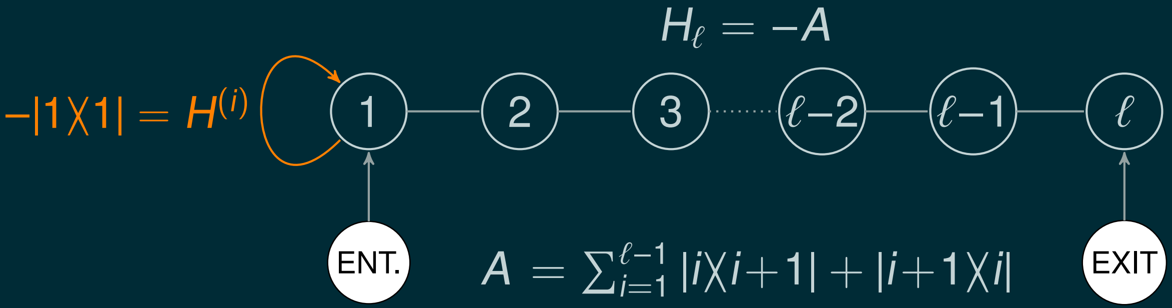

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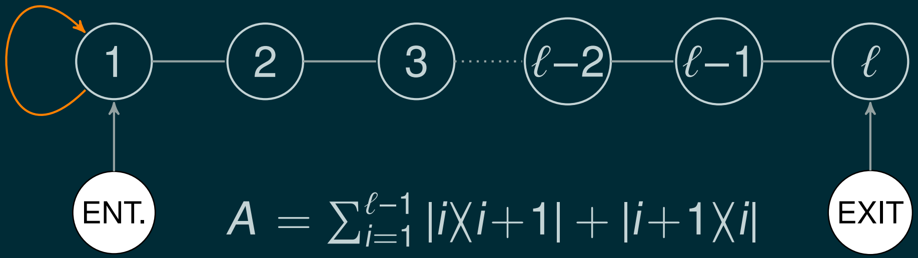

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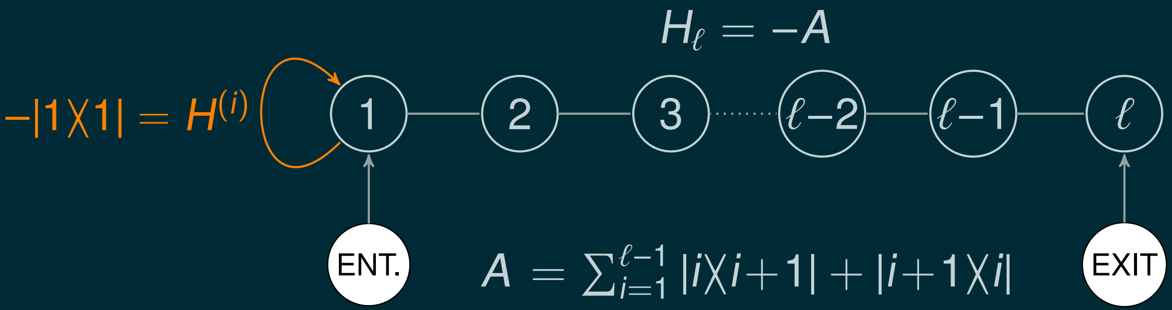

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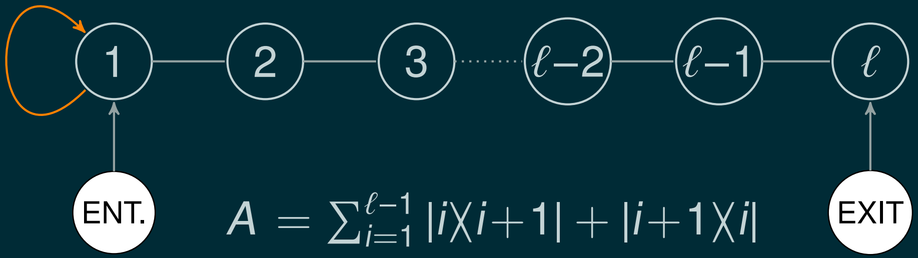

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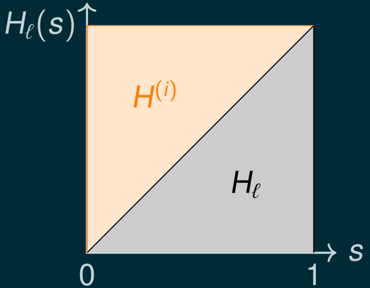

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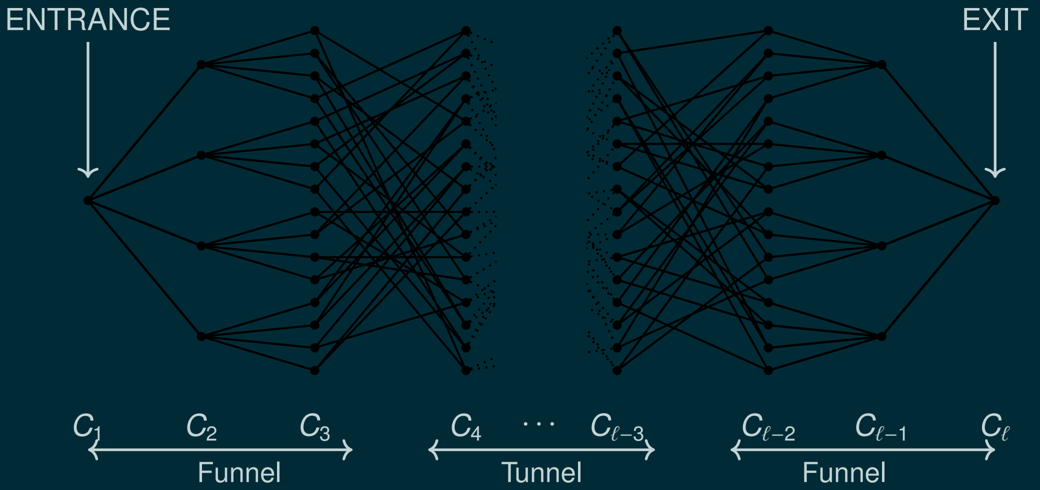

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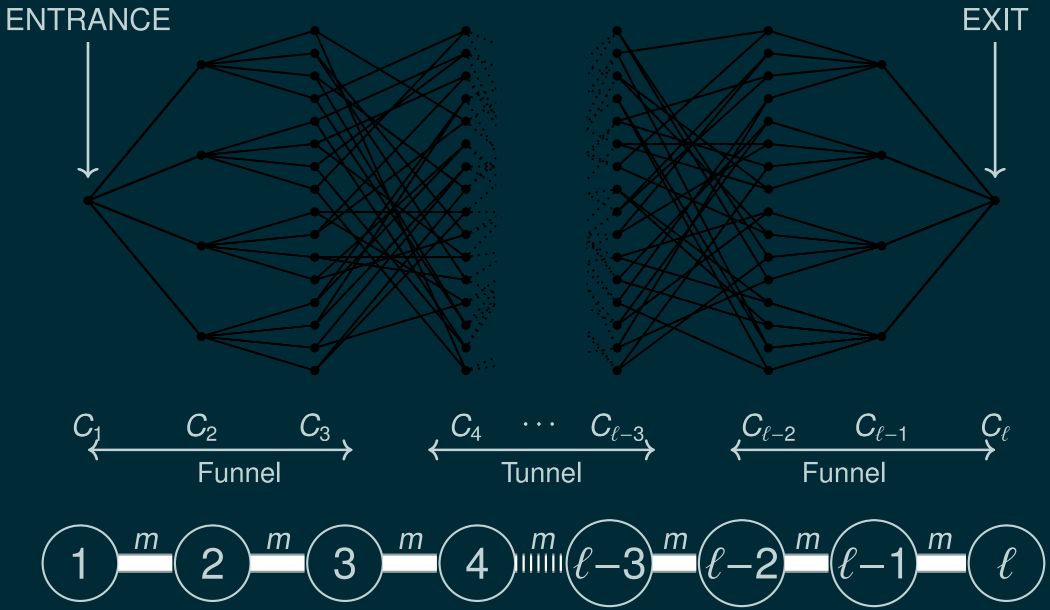

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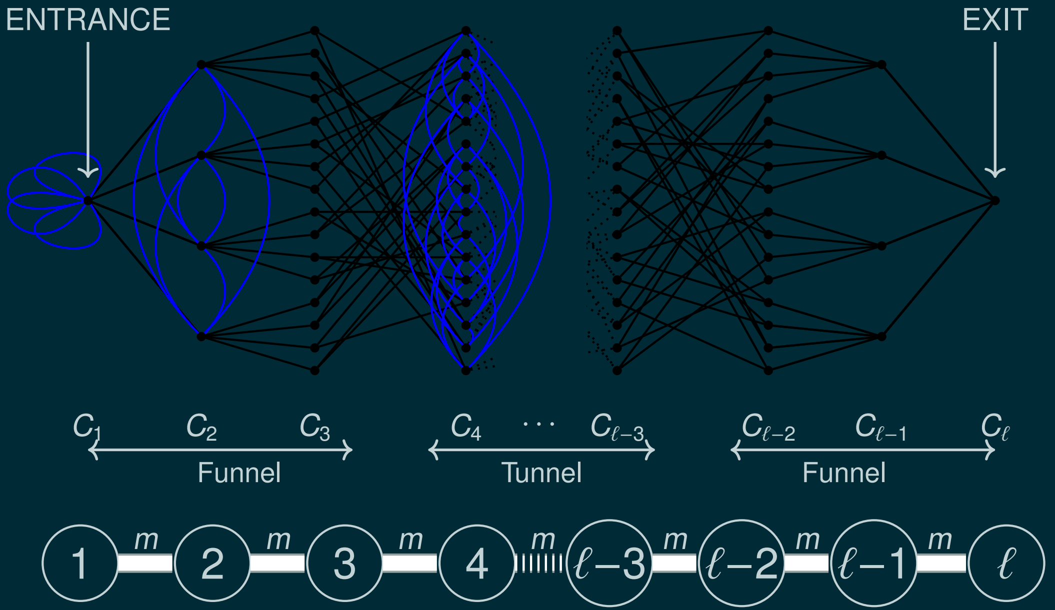

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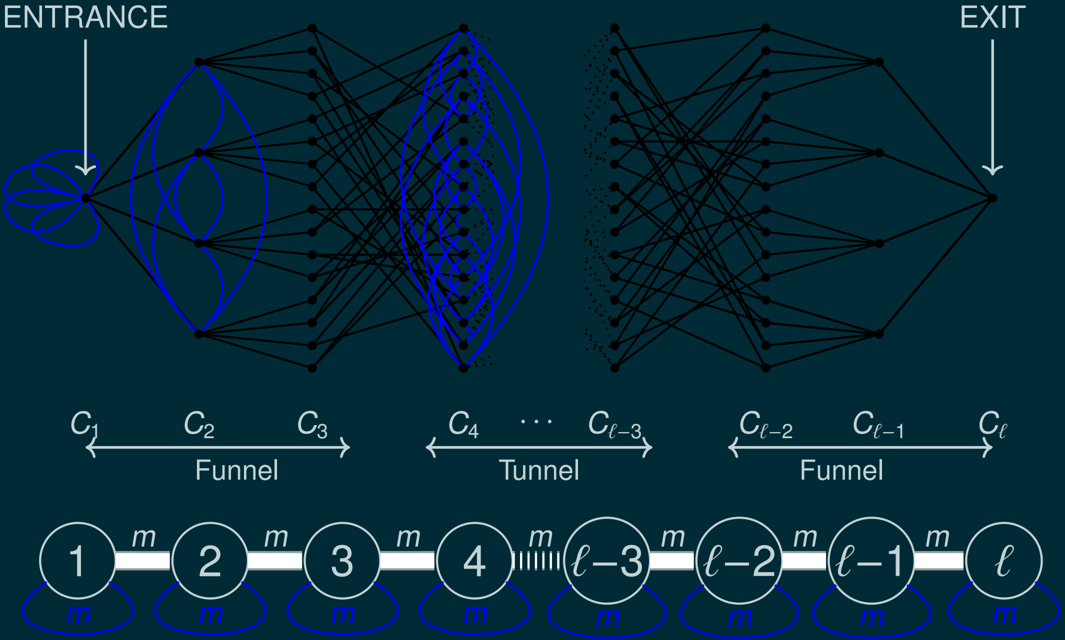

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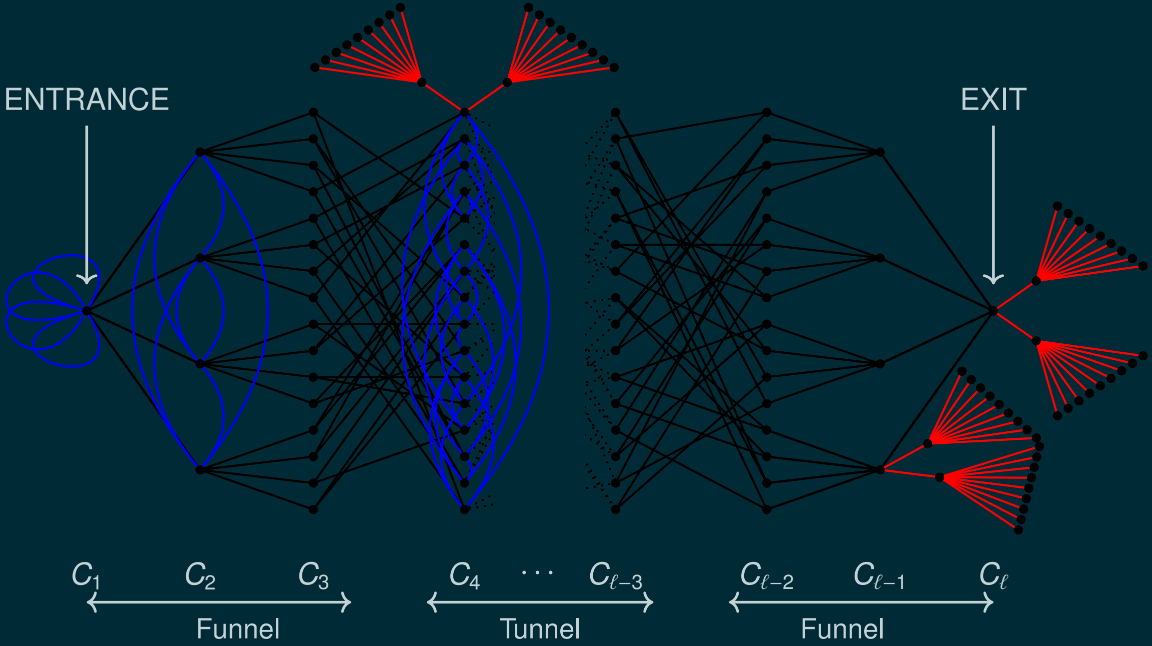

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- However, if the trees are $\mathrm{poly}(m)$ deep, then the ℓ_1 weight moves onto the trees!

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- Due to the random labeling of vertices no classical algorithm can navigate the decorated graph, and any classical algorithm will “get lost” spending an exponential amount of time in the “camouflage forest” before finding the EXIT.

Open questions

- Find a sign-problem-free Hamiltonian of practical interest providing a large speed-up!
- Are sign-problem-free Hamiltonians at least marginally easier in general?
  Is there a general simulation algorithm that works in time for example $\sqrt{2^n}$?
- What is the classical complexity of simulating adiabatic evolution for frustrated local Hamiltonians with no sign problem? (See, e.g., [Bringewatt & Jarret, 2020].)