Characterization of solvable spin models via graph invariants

Quantum **4**, 278 (2020); ArXiv:2003.05465

Adrian Chapman and Steven T Flammia

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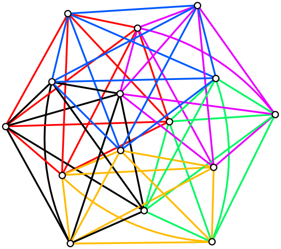
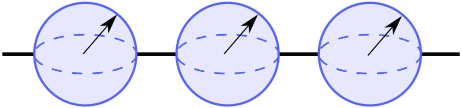
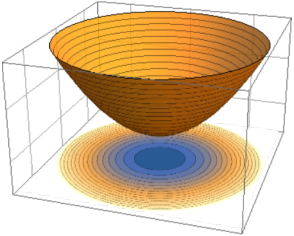


Exact Solutions for Spin Models

- Mapping to free-fermions is a workhorse method





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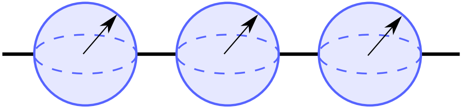

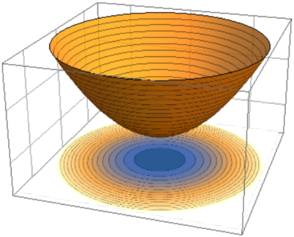

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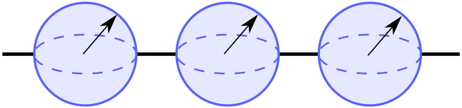

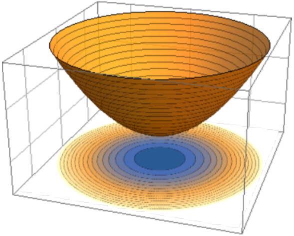

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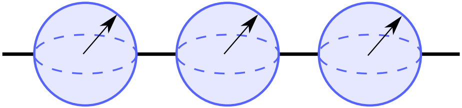
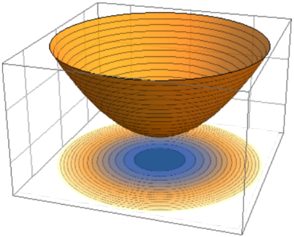
- Rich connection to complexity
  - Matchgate circuits[1−4]





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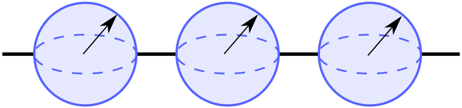

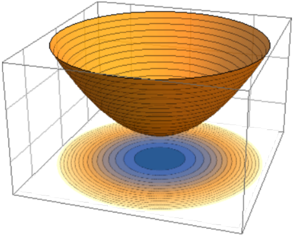

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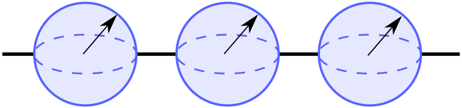

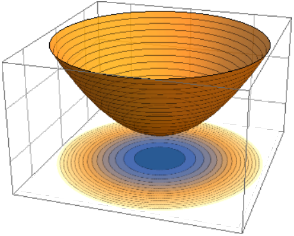

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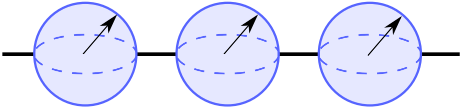

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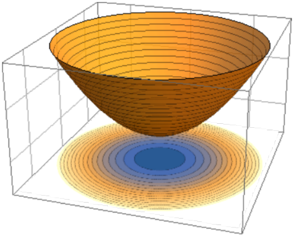

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Free Fermions

The following model is exactly solvable

$$H_{\text{solv}} = \sum_{j=1}^{n/2} (X_{2j-1}X_{2j} + Y_{2j}Y_{2j+1}) + \sum_{j=1}^{n} Z_j$$

Jordan-Wigner Transformation[11]

$$\begin{cases} \gamma_{2j-1} = Z^{\otimes(j-1)} \otimes X_j \otimes I^{\otimes(n-j)} \\ \gamma_{2j} = Z^{\otimes(j-1)} \otimes Y_j \otimes I^{\otimes(n-j)} \end{cases}$$

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Exact solution

$$e^{4\mathbf{w}} \cdot \mathbf{h} \cdot e^{-4\mathbf{w}} = \oplus_{j=1}^{n} \begin{pmatrix} 0 & -\lambda_j \\ \lambda_j & 0 \end{pmatrix}$$

$$e^{-\boldsymbol{\gamma}\cdot\mathbf{w}\cdot\boldsymbol{\gamma}^{\mathrm{T}}} H_{\text{solv}} e^{\boldsymbol{\gamma}\cdot\mathbf{w}\cdot\boldsymbol{\gamma}^{\mathrm{T}}} = 2\sum_j \lambda_j Z_j$$

$$E_{\boldsymbol{x}} = 2\sum_j (-1)^{x_j} \lambda_j$$

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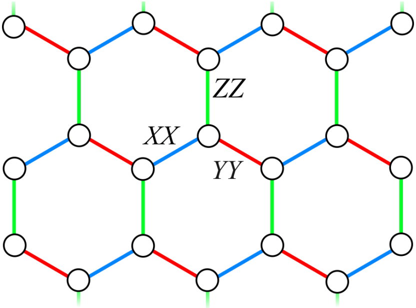
Another Example: Kitaev Honeycomb Model

$$H_{\mathrm{KHM}} = -J_x \sum_{x-\mathrm{links}} X_j X_k - J_y \sum_{y-\mathrm{links}} Y_j Y_k - J_z \sum_{z-\mathrm{links}} Z_j Z_k$$

- Compass model on honeycomb lattice.



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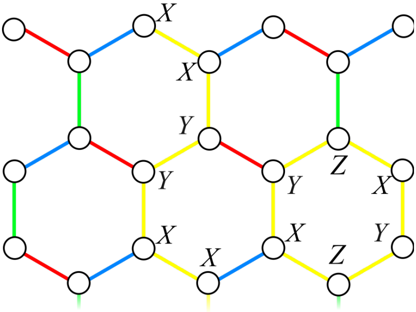

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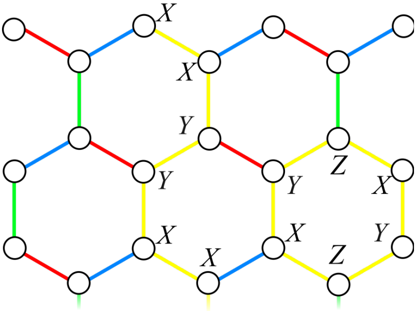
- For an $L_x \times L_y$ lattice, the effective Hilbert space contains $O(L_x L_y)$ qubits in a mutual eigenspace of the cycles.



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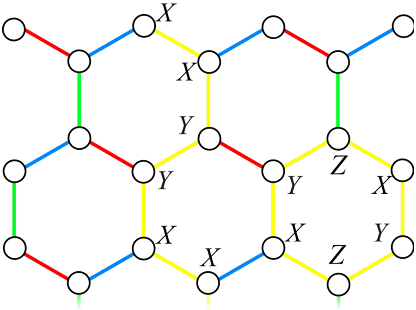
- Compass model on honeycomb lattice.
- Bonds on cycles multiply to constants of motion.
- For an $L_x \times L_y$ lattice, the effective Hilbert space contains $O(L_x L_y)$ qubits in a mutual eigenspace of the cycles.
- A free-fermion mapping is needed to complete the solution.



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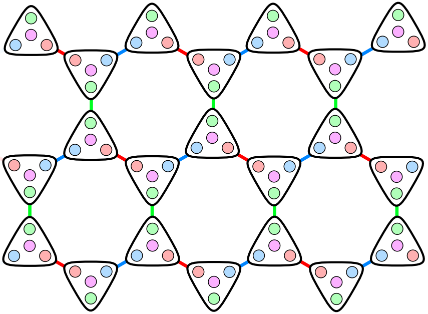
- Map each qubit to **four** fermions

$$\sigma_k^\alpha = i b_k^\alpha c_k$$

with a new symmetry at each vertex

$$D_j \equiv b_j^x b_j^y b_j^z c_j$$



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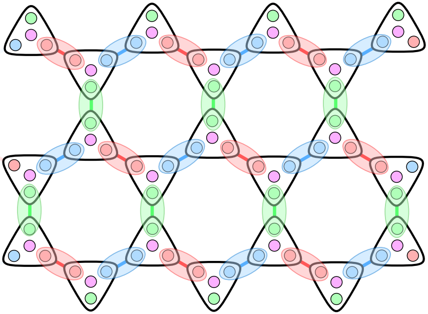

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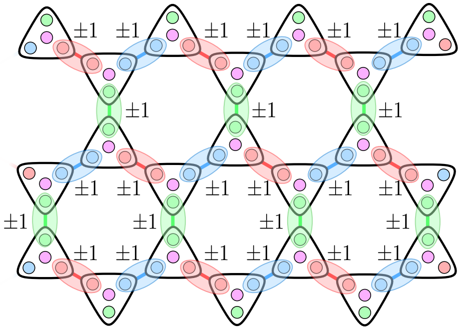
- Solve a free-fermion Hamiltonian in each sector



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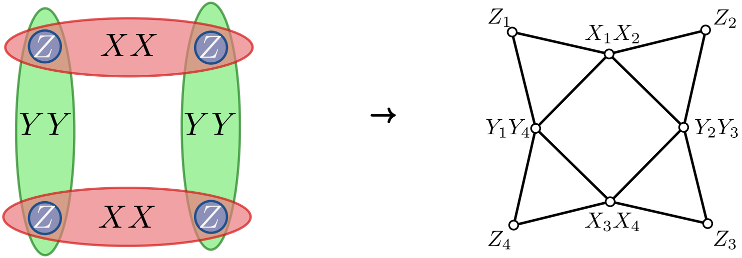
How to unify these approaches?

Definition (Frustration Graph)

Given a Hamiltonian H written in a specified Pauli basis, its **frustration graph**, $G(H) \equiv (V, E)$, has vertices corresponding to the Pauli terms in H. Vertices are neighboring in $G(H)$ if and only if their corresponding Paulis anticommute.

$$H_{\mathrm{solv}} = \sum_{j=1}^{n/2} \left(X_{2j-1} X_{2j} + Y_{2j} Y_{2j+1} \right) + \sum_{j=1}^{n} Z_j$$



When is a mapping to free fermions possible?

Given a general Hamiltonian in the Pauli basis

$$H = \sum_j h_j P_j$$

when can we define distinct quadratic fermion operators such that commutation relations are respected?

$$P_j \mapsto i\gamma_{j_1}\gamma_{j_2} \qquad \text{such that} \qquad P_j P_k = (-1)^{|(j_1,j_2)\cap(k_1,k_2)|} P_k P_j$$

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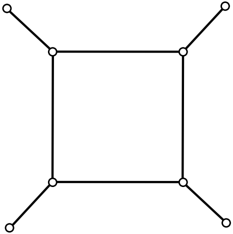
In graph theoretic terms...

When can we label vertices of the frustration graph $G(H)$ by subsets of size at most two, such that neighboring vertices intersect in exactly one element?

Line graphs!

Definition

The **line graph** of a *root graph* $R \equiv (V, E)$ is a graph $L(R) \equiv (E, F)$ whose vertices correspond to the edges of R such that two vertices are neighboring in $L(R)$ if the corresponding edges of R share a vertex.



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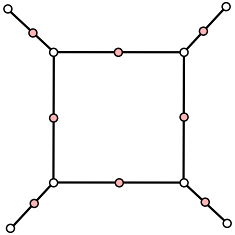

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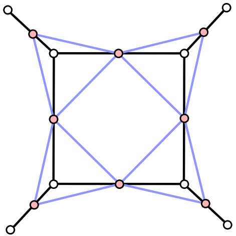

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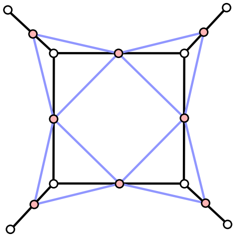

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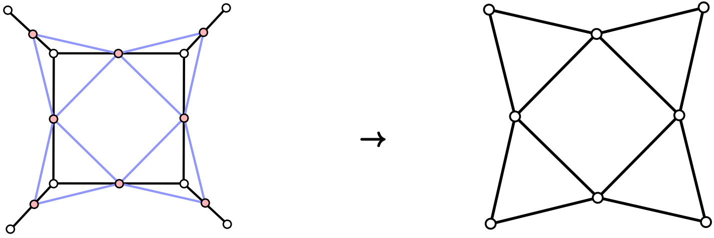

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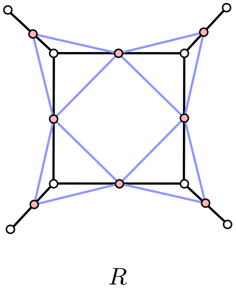

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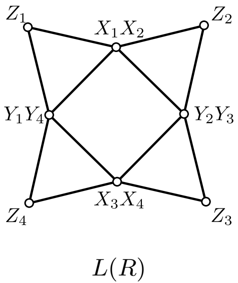

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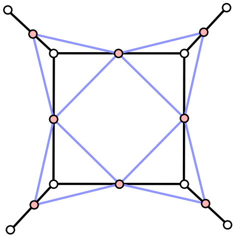
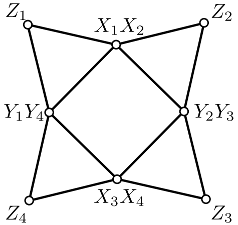


R

$\rightarrow$



$L(R)$

Definition (Krausz decomposition[13])

A graph is a line graph iff there exists an edge partition into cliques such that every vertex belongs to at most two cliques.

[13] J. Krausz, Mat. Fis. Lapok **50** (1943), 75-85

Fundamental Theorem

Theorem (Existence of free-fermion solution)

Given a Hamiltonian in the Pauli basis

$$H = \sum_j h_j P_j$$

an injective mapping

$$P_j \mapsto i\gamma_{j_1}\gamma_{j_2} \qquad \text{such that} \qquad P_j P_k = (-1)^{|(j_1,j_2)\cap(k_1,k_2)|} P_k P_j$$

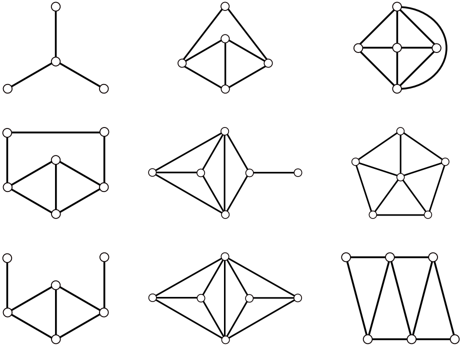
exists iff the frustration graph of H is the line graph $L(R)$ for some root graph R. The root graph R is the hopping graph of the fermions.

Proof sketch

- $\Rightarrow$ (Definitions coincide)
- $\Leftarrow$ If $G(H)$ is a line graph, associate a fermion to each clique in its Krausz decomposition, and give each Pauli the fermions corresponding to its cliques.

Characterization

- A graph is a line graph iff no subset of its vertices induces one of the following nine forbidden subgraphs[14].



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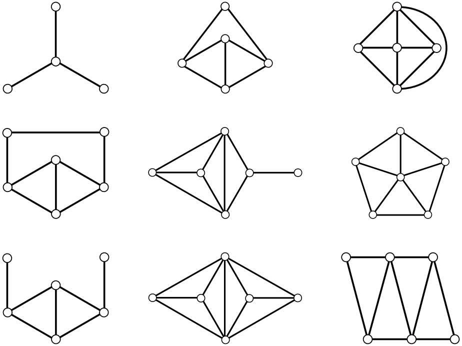

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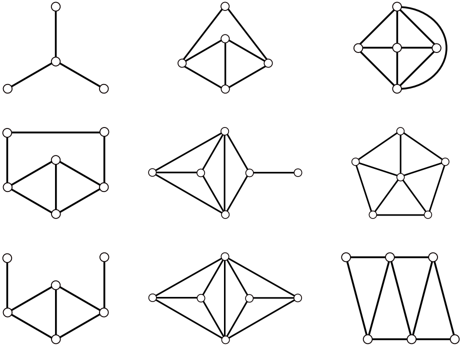

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- Line graphs can be recognized efficiently[15−17].

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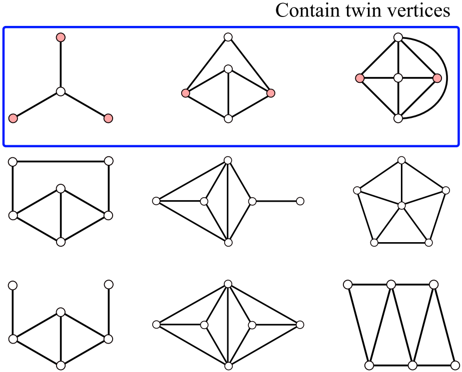

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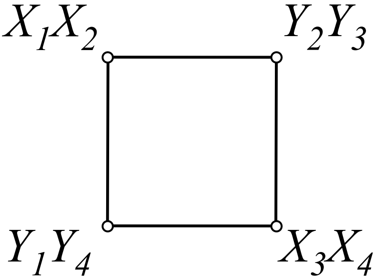

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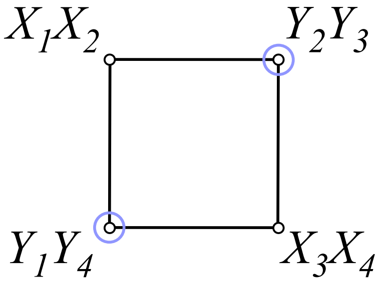

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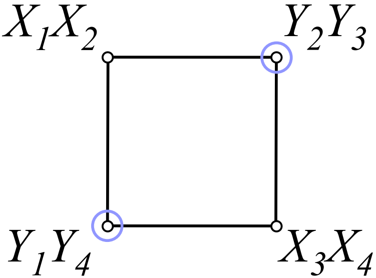
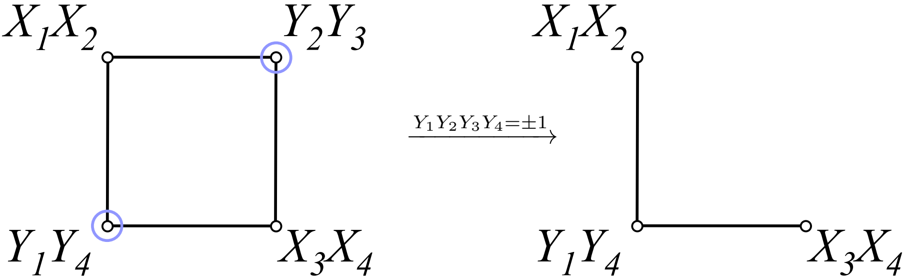


$$\xrightarrow{Y_1Y_2Y_3Y_4=\pm 1}$$

Twin Vertices



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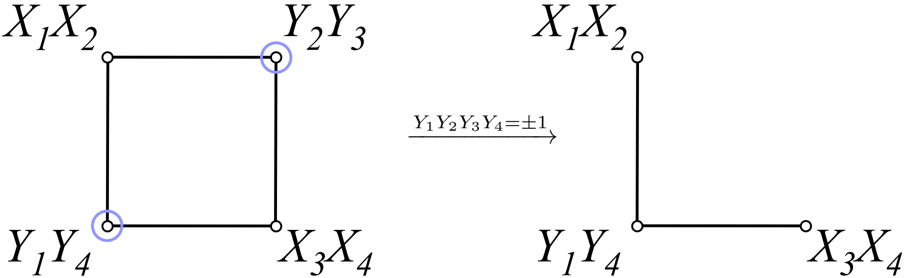

- If vertices have the same neighbors, then their product commutes with every term in the Hamiltonian, and they can be removed by fixing a symmetry.

Uniqueness

- Except for K_3, the root graph of any line graph is unique.

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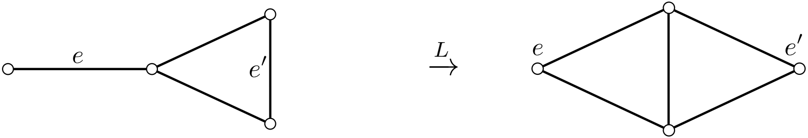

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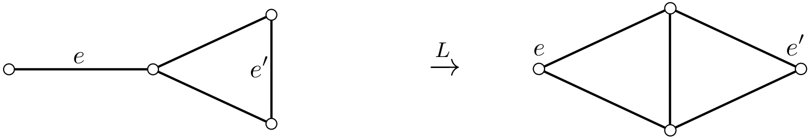

- This implies that for free-fermion Hamiltonians of more than four modes, any Clifford symmetry of the Hamiltonian is also a symmetry of the single-particle Hamiltonian.

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Root-Graph Symmetries

Given the Hamiltonian in the Pauli basis

$$H = \sum_j h_j P_j$$

what are products that commute with every term in the Hamiltonian?

$$[\prod_{j \in S} P_j, P_k] = 0 \qquad \forall k$$

These products correspond to elements in the binary kernel of the adjacency matrix of $G(H)$

(i) Twin vertices

(ii) Cycles in the root graph

(iii) Fermionic parity operator ($P = \gamma_1 \gamma_2 \ldots \gamma_{|V|}$)

Cycles and Parity

The adjacency matrix of a line graph $L(R) \equiv (E, F)$ with root $R \equiv (V, E)$ can be factorized as

$$\mathbf{A} = \mathbf{B}\mathbf{B}^{\mathrm{T}} \pmod 2$$

$\mathbf{B}$ is the root-graph incidence matrix

$$B_{ij} = \begin{cases} 1 & \text{if vertex j} \in \text{V belongs to edge i} \in \text{E} \\ 0 & \text{otherwise} \end{cases}$$

Graphical symmetries are vectors $\mathbf{v} \in \{0, 1\}^{\times |E|}$ in the binary kernel of $\mathbf{A}$

$$\mathbf{A} \cdot \mathbf{v} = \mathbf{0} \pmod 2$$

There are two cases

(i) $\mathbf{B}^{\mathrm{T}} \cdot \mathbf{v} = \mathbf{0} \ \Rightarrow \mathbf{v}$ is a subgraphs of even-degree (a cycle)

(ii) $\mathbf{B} \cdot \left(\mathbf{B}^{\mathrm{T}} \cdot \mathbf{v}\right) = \mathbf{0} \ \Rightarrow \mathbf{B}^{\mathrm{T}} \cdot \mathbf{v} = \mathbf{1}$ (fermionic parity operator)

Incorporating Symmetries

- The sign of a given term is changed by exchanging $j_1 \leftrightarrow j_2$ in the mapping

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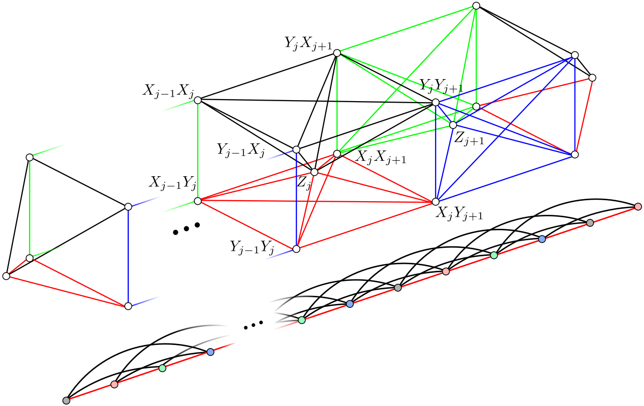
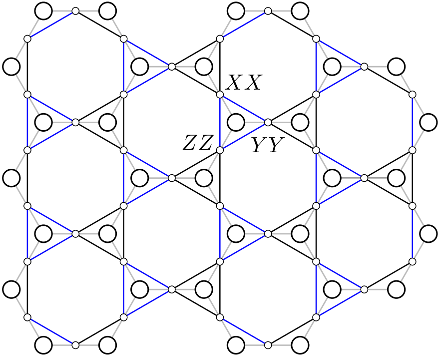
  3. For each edge not in the spanning tree, choose the orientation according to the sign of the uniquely associated independent cycle.
- If the parity operator is proportional to the identity (up to a product of cycles) in the spin model, we may need to fix a parity symmetry of the free-fermion Hamiltonian.

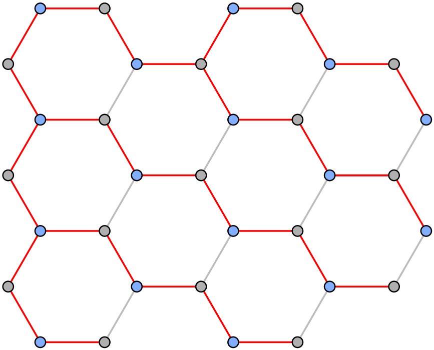
Nearest-Neighbor 1-d Model

$$H = \sum_{j=1}^{n-1} \sum_{\alpha,\beta \in \{x,y\}} \mu^{j}_{\alpha\beta} \sigma^{\alpha}_{j} \otimes \sigma^{\beta}_{j+1} + \sum_{j=1}^{n} \nu_j Z_j$$



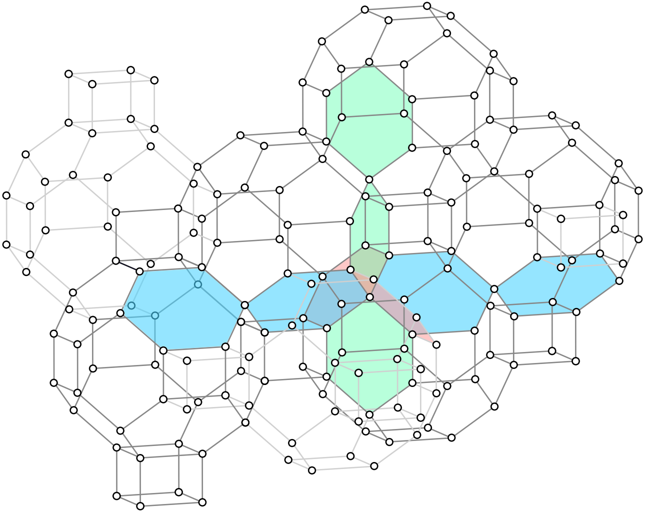
Kitaev Honeycomb Model





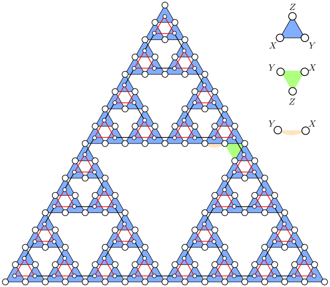
- The frustration graph is the line graph of the honeycomb graph.
- Orientation of edges outside of the spanning tree specifies a symmetry sector of the plaquettes.

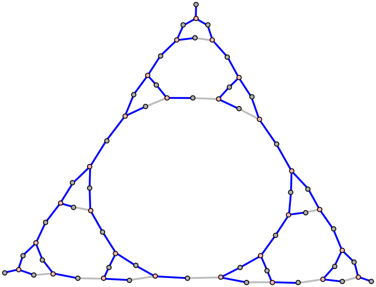
Frustrated Hexagonal Gauge 3D Color Code



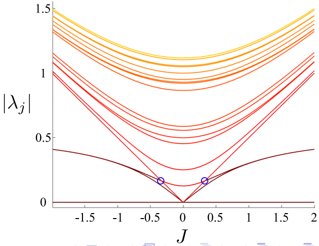
[19] T. Jochym-O'Connor, S. Roberts, S. Bartlett, and J. Preskill, QEC 2019

Sierpinski-Hanoi Model





- Frustration graph describes allowed transitions in the Towers of Hanoi.
- Model encodes logical qubits at a constant asymptotic rate of $\frac{11}{18}$.
- An excited-state degeneracy emerges upon introducing a local field.



Summary

- We give a graph-theoretic characterization of a wide class of fermion-solvable models.
  - Solvable when the frustration graph is a line graph.
  - Graphical symmetries correspond to cycles and parity.
- Still not completely characterized
  - Beyond generator-to-generator mappings
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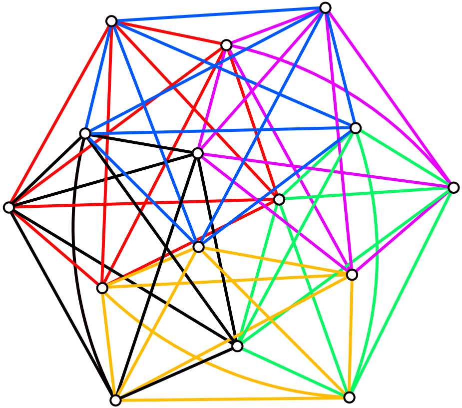

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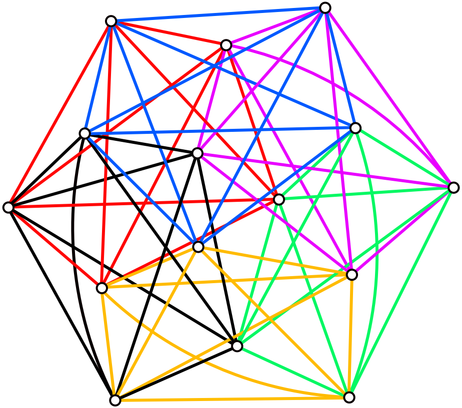

Thanks!

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